Solutions 11

Exercise 11.1. Let $A = (a_1|\cdots|a_n)$ be an $m \times n$ matrix of rank r over some field $\mathbb{F}$. Since A is of rank r, the dimension of its image when A is viewed as a linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$ is r. Let $\{\beta_1, \ldots, \beta_r\}$ be a basis of ImA, for $\beta_i \in \mathbb{F}^m$. Each column a_i of A can be viewed as an element of ImA and can thus be written as a linear combination of the basis vectors as follows:

$$a_i = \sum_{j=1}^{r} a_{ji}\beta_j.$$

We can thus write

$$A = \begin{pmatrix} \beta_1 & \cdots & \beta_r \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & & a_{rn} \end{pmatrix}.$$

Instead of being represented by mn entries, A can thus be represented by $r(m+n)$ entries, which is a significant gain when r is small enough.

Exercise 11.2.

1. Let $A = \text{diag}(1/x_1, \ldots, 1/x_n)$ and let $B := Z_t$ be the $t \times t$ shift operator

$$B_t = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

   Then

$$AD^iV_{n,t} - D^iV_{n,t}B = \begin{pmatrix} y_1^i/x_1 \\ y_2^i/x_2 \\ \vdots \\ y_n^i/x_n \end{pmatrix} \cdot \underbrace{(1, 0, \ldots, 0)}_{t}.$$

2. We define a displacement operator $\nabla_{A,B}$ with $A = \text{diag}(1/x_1, \ldots, 1/x_n)$ as before, and B a block-diagonal matrix with diagonal blocks $B_e, B_{e+(k-1)}, \cdots, B_{e+\ell(k-1)}$ and zeroes everywhere else. Then

$$AX - XB = \left(\begin{array}{cccc|cccc|c|cccc} y_1^\ell/x_1 & 0 & \cdots & 0 & y_1^{\ell-1}/x_1 & 0 & \cdots & 0 & \cdots & 1/x_1 & 0 & \cdots & 0 \\ y_2^\ell/x_2 & 0 & \cdots & 0 & y_2^{\ell-1}/x_2 & 0 & \cdots & 0 & \cdots & 1/x_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & & & & \vdots & & & \\ y_n^\ell/x_n & 0 & \cdots & 0 & y_n^{\ell-1}/x_n & 0 & \cdots & 0 & \cdots & 1/x_n & 0 & \cdots & 0 \end{array}\right)$$

$$= \begin{pmatrix} y_1^\ell/x_1 & y_1^{\ell-1}/x_1 & \cdots & 1/x_1 \\ y_2^\ell/x_2 & y_2^{\ell-1}/x_2 & \cdots & 1/x_2 \\ \vdots & \vdots & \ddots & \vdots \\ y_n^\ell/x_n & y_n^{\ell-1}/x_n & \cdots & 1/x_n \end{pmatrix} \cdot \left(\begin{array}{cccc|cccc|c|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & & & & \vdots & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \end{array}\right)$$

which has rank $\ell+1$ and is such that we can recover $x_1,\cdots,x_n$ and $y_1,\cdots,y_n$, i.e., $\nabla_{A,B}$ is indeed bijective.

Exercise 11.3. Let

$$A=\begin{pmatrix}0&1&&&\\&0&1&&\\&&\ddots&\ddots&\\&&&0&1\\&&&&0\end{pmatrix},\qquad B=\begin{pmatrix}0&&&&\\1&0&&&\\&\ddots&\ddots&&\\&&1&0&\\&&&1&0\end{pmatrix}.$$

Then

$$AT=\begin{pmatrix}a_2&\cdots&a_{t+1}\\a_3&\cdots&a_{t+2}\\\vdots&&\\a_t&\cdots&a_{2t-1}\\0&\cdots&0\end{pmatrix}\text{ and }TB=\begin{pmatrix}a_2&\cdots&a_t&0\\a_3&\cdots&a_{t+1}&0\\\vdots&&&\\a_{t+1}&\cdots&a_{2t-1}&0\end{pmatrix},$$

so that

$$AT-TB=\begin{pmatrix}0&\cdots&0&a_{t+1}\\&&&\vdots\\0&\cdots&0&a_{2t-1}\\a_{t+1}&\cdots&a_{2t-1}&0\end{pmatrix},$$

which has rank 2.

Exercise 11.4. Let $L=\begin{pmatrix}\ell_1&&\\&\ddots&\\&&\ell_n\end{pmatrix}$ and $U=\begin{pmatrix}u_1&&\\&\ddots&\\&&u_n\end{pmatrix}$. Then

$$LC-CU=\begin{pmatrix}\frac{\ell_1}{x_1-y_1}&\frac{\ell_1}{x_1-y_2}&\cdots&\frac{\ell_1}{x_1-y_n}\\\frac{\ell_2}{x_2-y_1}&\frac{\ell_2}{x_2-y_2}&\cdots&\frac{\ell_2}{x_2-y_n}\\\vdots&\vdots&\ddots&\vdots\\\frac{\ell_n}{x_n-y_1}&\frac{\ell_n}{x_n-y_n}&\cdots&\frac{\ell_n}{x_n-y_n}\end{pmatrix}-\begin{pmatrix}\frac{u_1}{x_1-y_1}&\frac{u_2}{x_1-y_2}&\cdots&\frac{u_n}{x_1-y_n}\\\frac{u_1}{x_2-y_1}&\frac{u_2}{x_2-y_2}&\cdots&\frac{u_n}{x_2-y_n}\\\vdots&\vdots&\ddots&\vdots\\\frac{u_1}{x_n-y_1}&\frac{u_2}{x_n-y_n}&\cdots&\frac{u_n}{x_n-y_n}\end{pmatrix},$$

so that

$$C_{ij}=\frac{\ell_i-u_j}{x_i-y_j}.$$

We see that a choice of $\ell_i=x_i^2$, $u_j=y_j^2$ yields the matrix whose (i,j)-th entry is x_i+y_j. But this matrix is of rank 2 and given by

$$\begin{pmatrix}x_1&1\\x_2&1\\\vdots&\\x_n&1\end{pmatrix}\cdot\begin{pmatrix}1&1&\cdots&1\\y_1&y_2&\cdots&y_n\end{pmatrix}.$$