Exercise Sheet 11

Exercise 11.1. Show that any $m \times n$ matrix of rank r can be written as a product of an $m \times r$ matrix by an $r \times n$ matrix. Conclude that low-rank matrices have a "compact" representation.

Exercise 11.2. Recall Sudan's list-decoding algorithm: given n pairs of points (x_i, y_i), the problem was to solve the system

$$X \begin{pmatrix} A_\ell \\ A_{\ell-1} \\ \vdots \\ A_0 \end{pmatrix} = 0.$$

Here

$$X = \left(D^\ell V_{n,e} \mid D^{\ell-1} V_{n,e+(k-1)} \mid \cdots \mid D V_{n,e+(\ell-1)(k-1)} \mid V_{n,e+\ell(k-1)} \right),$$

$$D = \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_n \end{pmatrix}, \quad V_{n,m} = \begin{pmatrix} 1 & x_1 & \cdots & x_1^{t-1} \\ 1 & x_2 & \cdots & x_2^{t-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{t-1} \end{pmatrix},$$

and $A_0, \ldots, A_\ell$ are column vectors of appropriate dimensions, for integers e, k, ℓ depending on the code parameters.

1. Find matrices A and B such that $A\left(D^i V_{n,t}\right) - \left(D^i V_{n,t}\right) B$ has rank 1.
2. Find a displacement operator for the matrix X such that X has displacement rank $\ell + 1$ with respect to this operator.

Exercise 11.3. Let T be a $t \times t$ Toeplitz matrix, defined as

$$T = \begin{pmatrix} a_1 & a_2 & \cdots & a_t \\ a_2 & a_3 & \cdots & a_{t+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_t & a_{t+1} & \cdots & a_{2t-1} \end{pmatrix}.$$

Find matrices A and B such that $AT - TB$ has rank 2.

Exercise 11.4. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be two sets of elements in a field $\mathbb{F}$, and suppose that for all i and j we have $x_i \neq y_j$. The matrix $C = (1/(x_i - y_j))_{i,j}$ is called a *Cauchy matrix*. Find two diagonal matrices L and U such that C has displacement rank 2 with respect to $\nabla_{L,U}$.