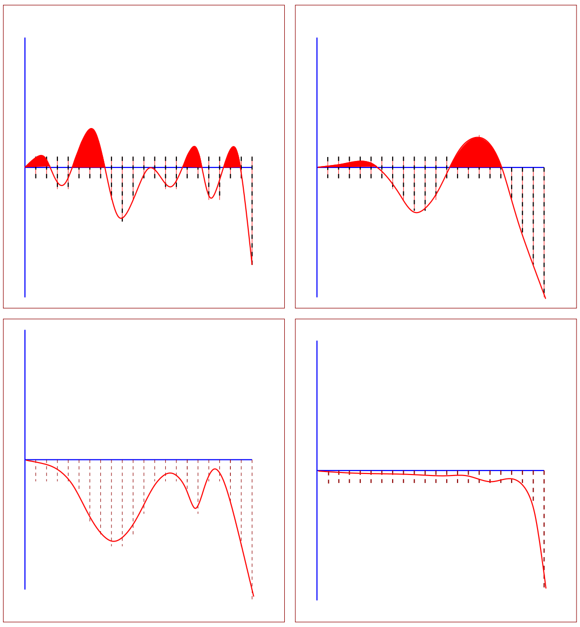
Design of Erasure Codes with Differential Evolution



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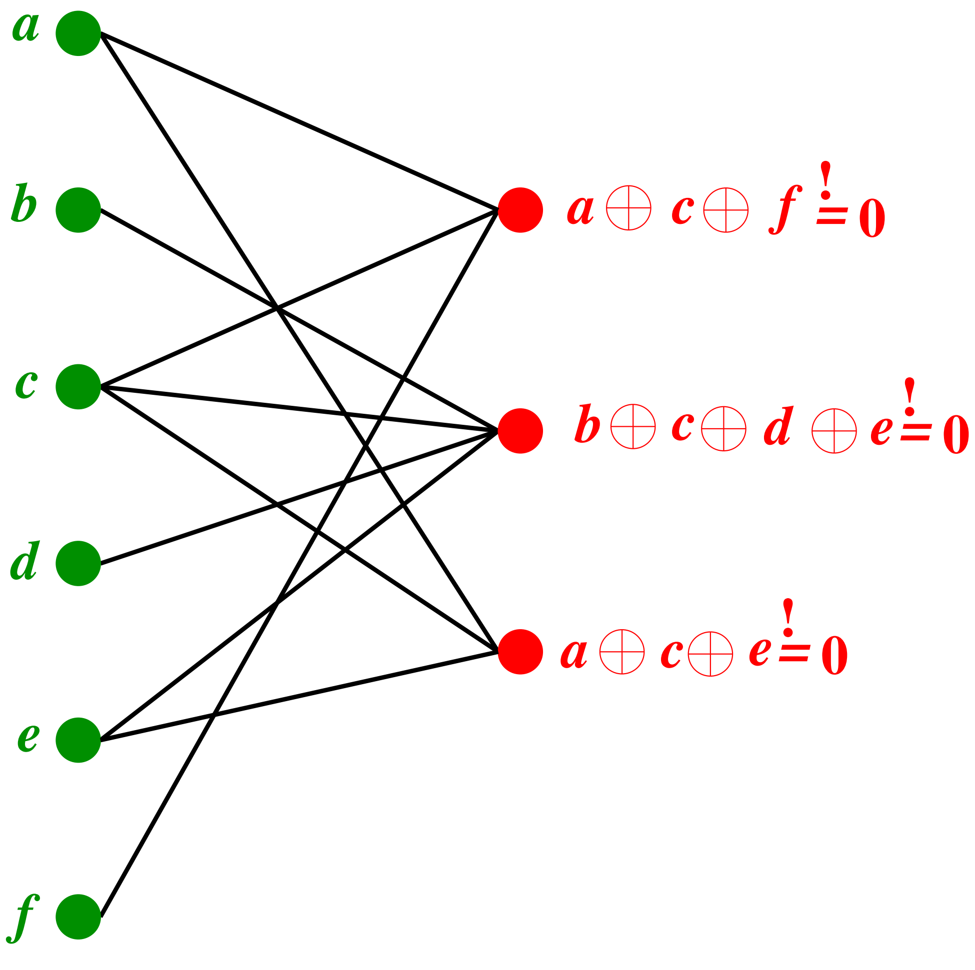
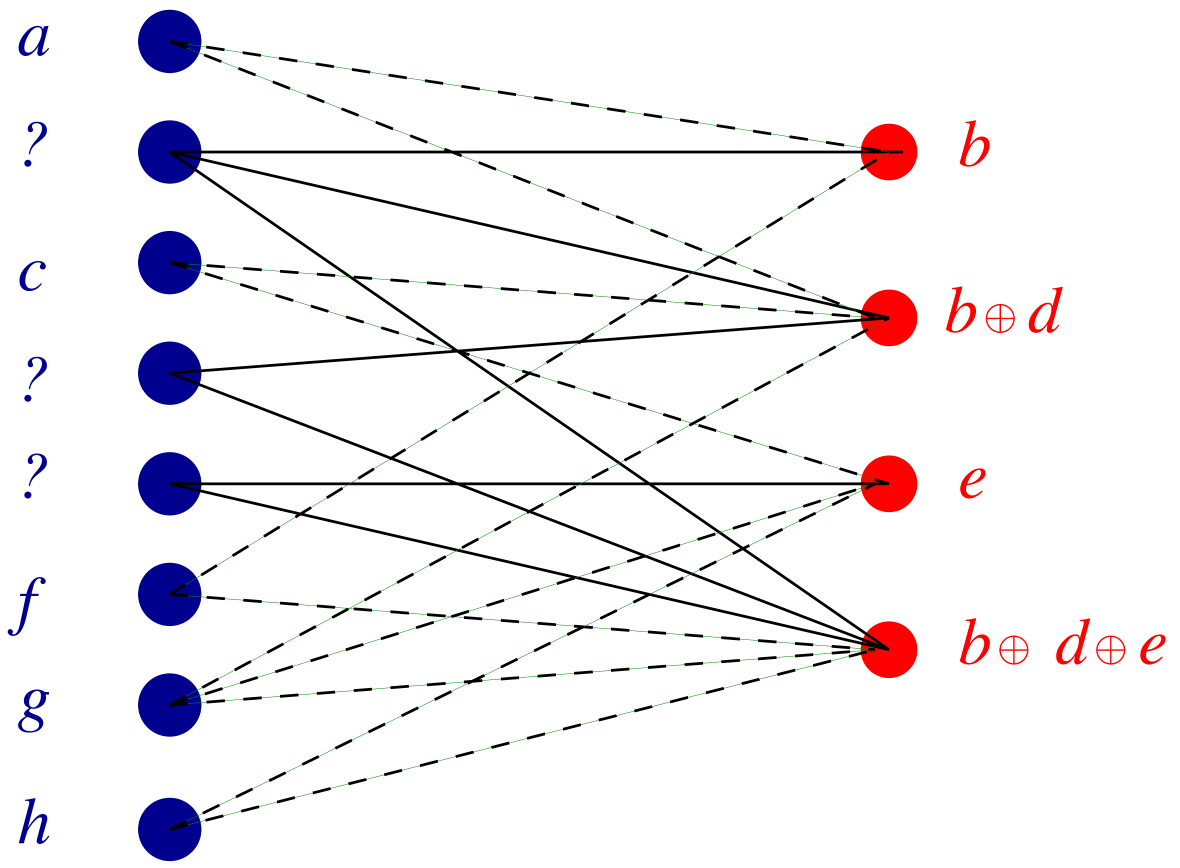
Bell Labs Innovations

Low-Density Codes



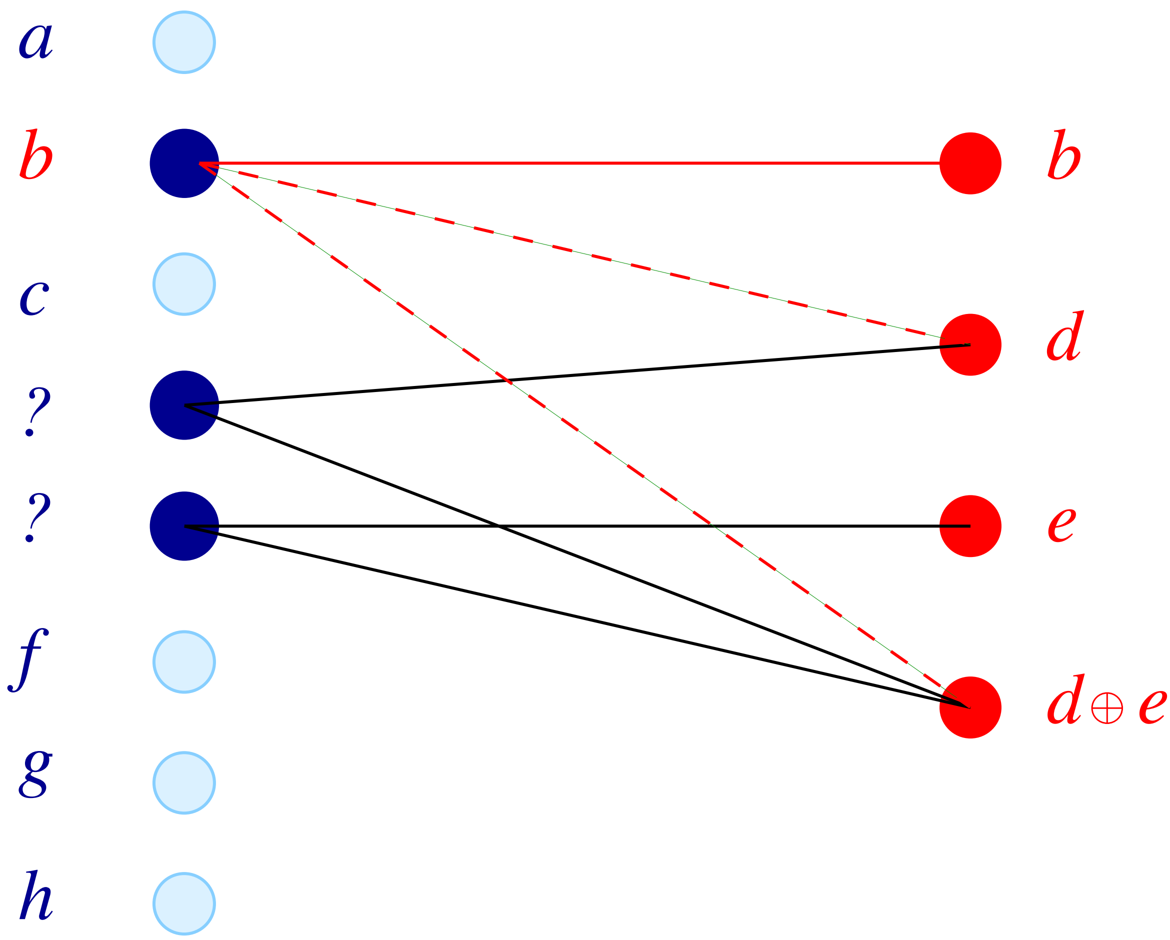
Decoding on the Erasure Channel

Stage 1: Direct Recovery



Decoding on the Erasure Channel

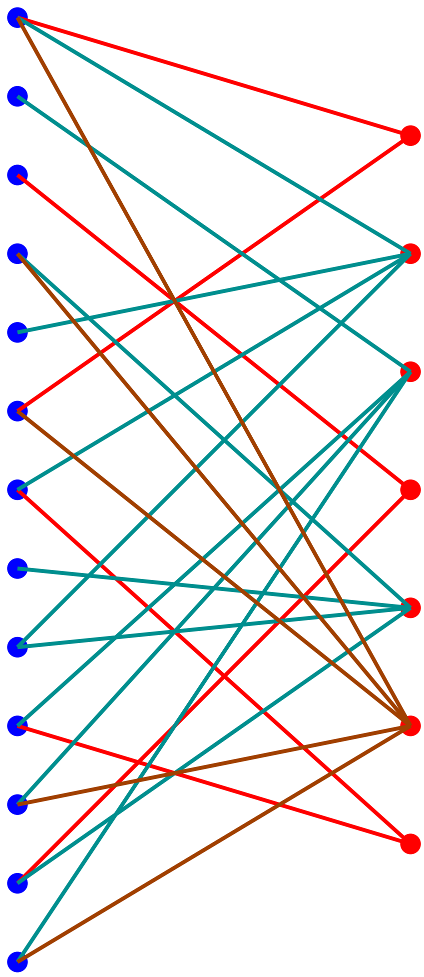
Stage 2: Substitution Recovery



Edge Fractions

λ_i fraction of edges of left degree i, $\lambda(x) = \sum_i \lambda_i x^{i-1}$.

ρ_i fraction of edges of right degree i, $\rho(x) = \sum_i \rho_i x^{i-1}$.



 # edges: 23
 # edges right degree 2: 6
 # edges right degree 4: 12
 # edges right degree 5: 5

$$\rho(x) = \frac{6}{23} x + \frac{12}{23} x^3 + \frac{5}{23} x^4$$

Successful Decoding

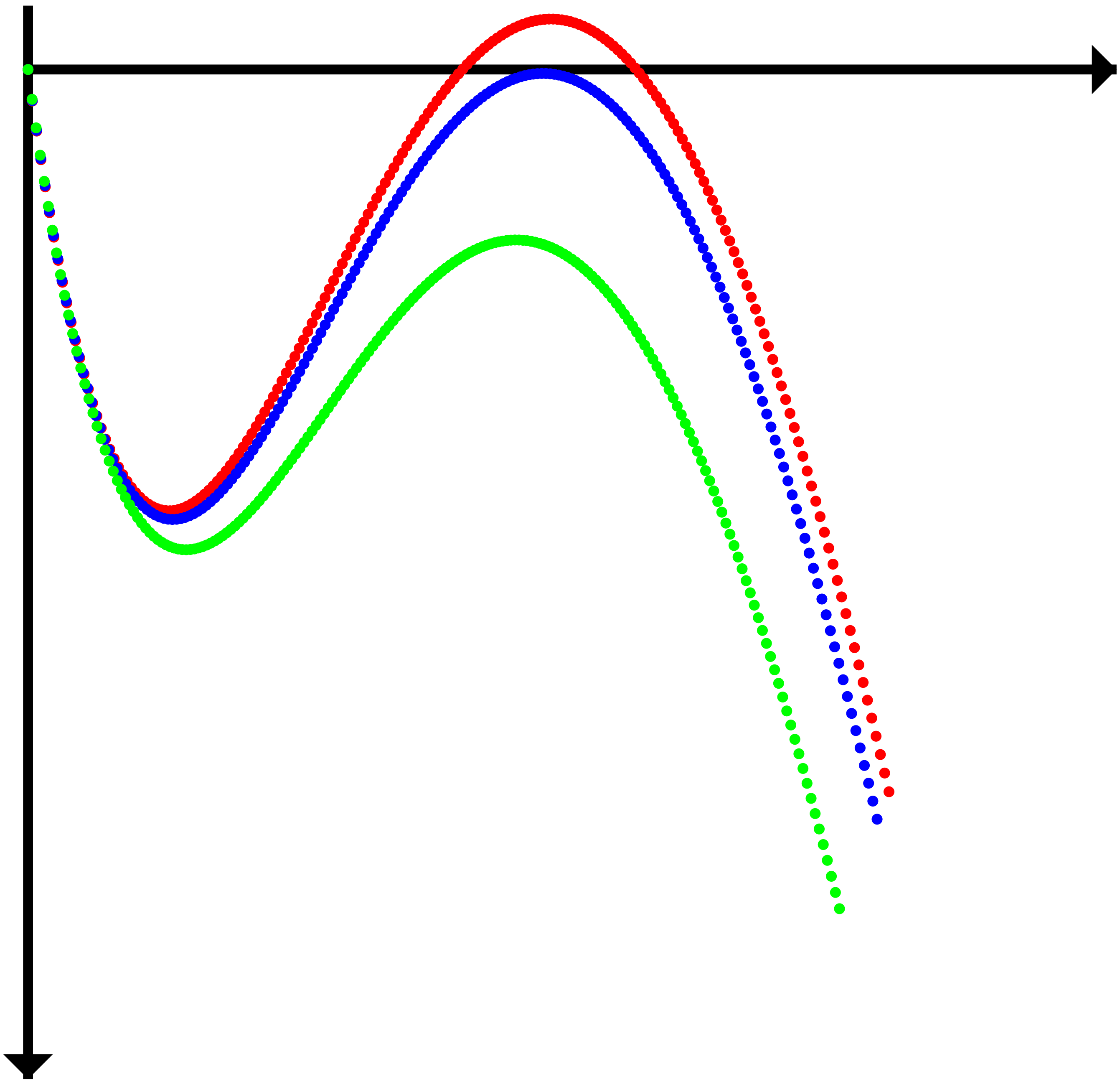
Successful Decoding possible with high probability if and only if

$$\delta\lambda\left(1-\rho(1-x)\right) < x$$

for all $x \in (0, \delta)$.

δ is original fraction or erasures.

(Luby, Mitzenmacher, Shokrollahi, Spielman, Stemann).



Optimization Problem

Fix rate R.

Maximize δ subject to

$$\delta\lambda(1-\rho(1-x)) < x$$

for $x \in (0,\delta)$ for some $\lambda(x)$ and $\rho(x)$ such that

$$1 - \frac{\int_0^1 \rho(x)\mathrm{d}x}{\int_0^1 \lambda(x)\mathrm{d}x} = 1 - R.$$

Bounds:

1. $\delta \leq 1 - \int_0^1 \rho(x)\mathrm{d}x / \int_0^1 \lambda(x)\mathrm{d}x =$ 1-R.

2. $\delta \leq \widehat{\delta}$ where

$$\widehat{\delta} = (1-R)(1-(1-\widehat{\delta})^a), \quad a = \frac{1}{\int_0^1 \rho(x)\mathrm{d}x}.$$

Linear programming

Discretize the interval $(0,1)$: $x_1, x_2, \ldots, x_n$.

Fix $\rho(x)$.

Compute $\lambda_2, \lambda_3, \ldots, \lambda_d$ with

$$\lambda_2 \rho(1-x_i) + \cdots + \lambda_d \rho(1-x_i)^d < \frac{x_i}{\delta}$$

for $i = 1, \ldots, d$. (Linear programming.)

Fix $\lambda(x)$.

Use dual condition

$$\rho(1 - \delta\lambda(1-x)) > x$$

on $(0,1)$.

Iterate!

Results: Linear Programming

$\delta = 0.4886$.

$$\lambda(x) = 0.11286x + 0.06153x^2 + 0.12936x^3 + 0.25559x^7 + 0.24226x^8 + 0.02550x^{24} + 0.17290x^{25}$$

$$\rho(x) = 0.17263x^5 + 0.28658x^6 + 0.14805x^{28} + 0.39274x^{29}$$

$\hat{\delta} = 0.4998$, $\delta/\hat{\delta} = 0.9775$.

$\delta = 0.4946$.

$$\lambda(x) = 0.196050x + 0.257821x^2 + 0.191453x^8 + 0.046831x^9 + 0.063126x^{23} + 0.059209x^{24} + 0.060652x^{62} + 0.124858x^{63}$$

$$\rho(x) = 0.820342x^8 + 0.177571x^9 + 0.002087x^{199}$$

$\hat{\delta} = 0.4992$, $\delta/\hat{\delta} = 0.9903$.

Problems with Linear Programming

1. Average degrees are always fixed.

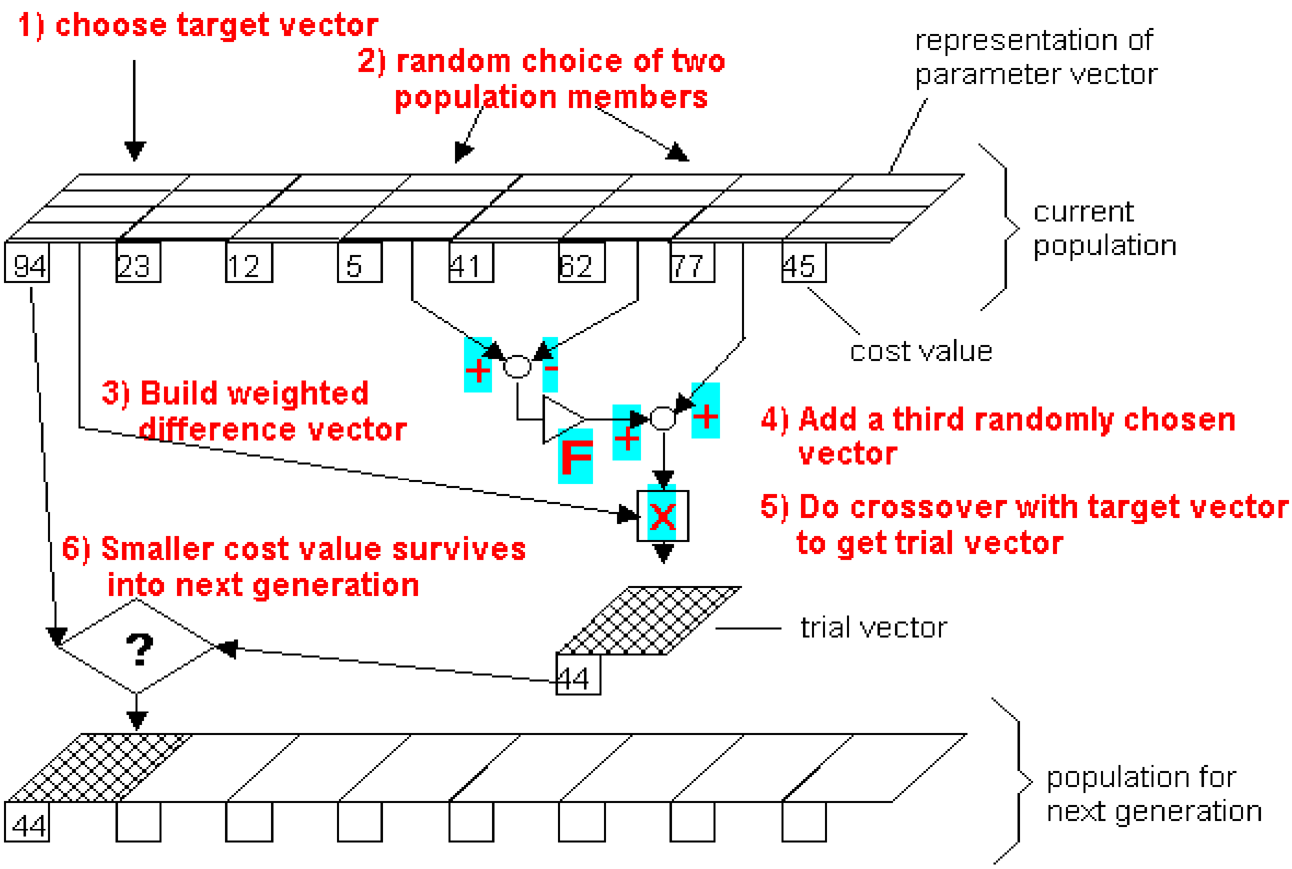
2. Sensitive to starting distribution.

3. Degrees are huge.

4. Procedure does not cover all possible degree distributions.

Differential Evolution

- Continuous optimizer.
- Combines genetic algorithms and gradient methods.
- Extremely robust.

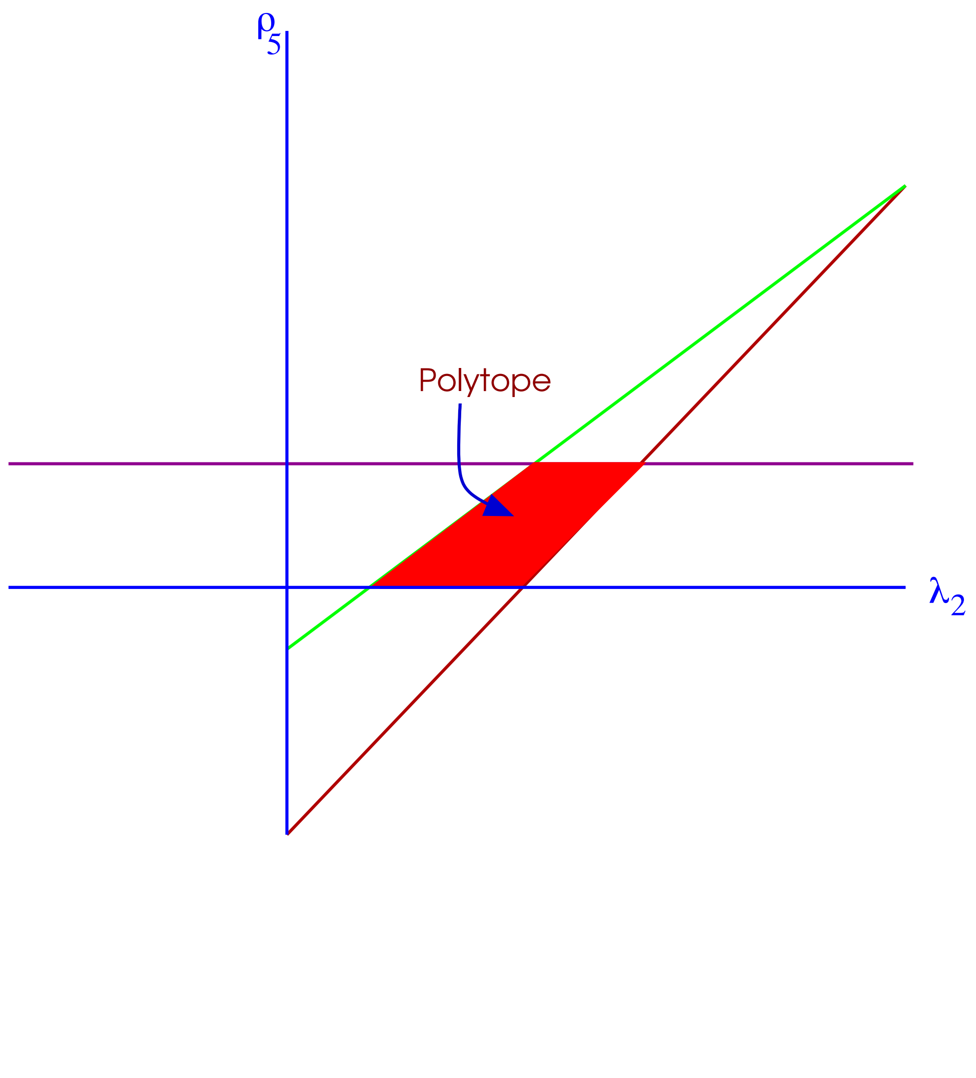


Admissibility Polytope

$$\frac{\lambda_2}{2} + \frac{\lambda_9}{9} + \frac{\lambda_{10}}{10} = \frac{\rho_5}{5} + \frac{\rho_6}{6}$$

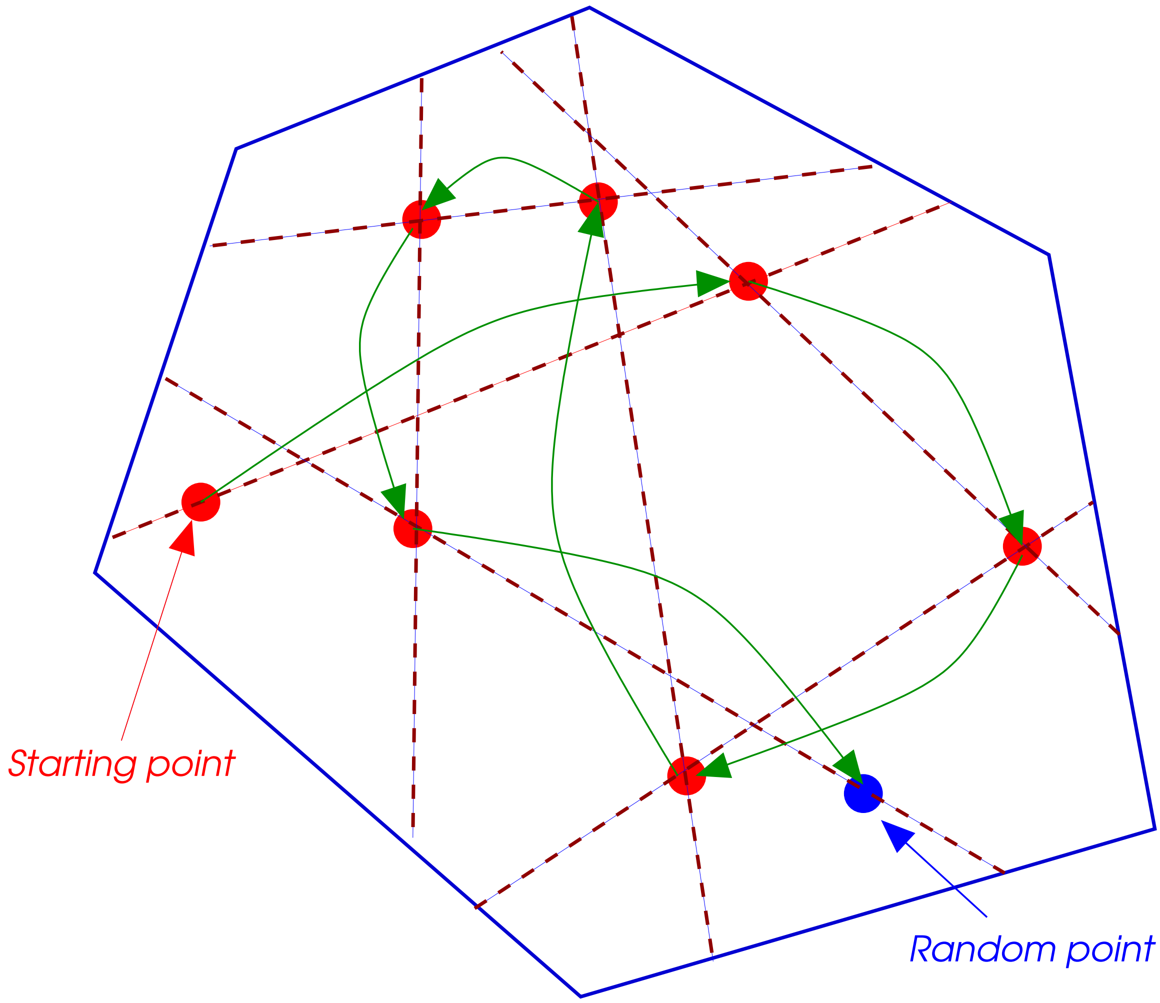
$$\lambda_2 + \lambda_9 + \lambda_{10} = 1$$
$$\rho_5 + \rho_6 = 1$$

$$\lambda_2, \lambda_9, \lambda_{10}, \rho_5, \rho_6 \geq 0.$$



Queen's Move

Sampling points in the admissibility polytope uniformly at random (conjecturally).



“Real” Degrees

Choice of the degrees occurring in $\lambda(x)$ and $\rho(x)$ is crucial.

Modified:

- $\tilde{\lambda}(x) = \sum_i \lambda_i x^{\alpha_i}, \ \alpha_i \in \mathbf{R}$.

- $\tilde{\rho}(x) = \sum_i \rho_i x^{\beta_i}, \ \beta_i \in \mathbf{R}$.

$$x^{\alpha} \quad \mapsto \quad a x^{\lfloor \alpha \rfloor} + b x^{\lceil \alpha \rceil}$$

$$\begin{aligned} a + b &= 1 \\ \frac{a}{\lfloor \alpha \rfloor} + \frac{b}{\lceil \alpha \rceil} &= \frac{1}{\alpha}. \end{aligned}$$

Results: Differential Evolution

$\delta = 0.4939$.

$$\lambda(x) = 0.29730x + 0.17495x^2 + 0.24419x^5 + 0.28353x^{19}$$

$$\rho(x) = 0.33181x^6 + 0.66818x^7$$

$\widehat{\delta} = 0.4974$, $\delta/\widehat{\delta} = 0.9929$.

$\delta = 0.4955$.

$$\lambda(x) = 0.26328x + 0.18020x^2 + 0.27000x^6 + 0.28649x^{29}$$

$$\rho(x) = 0.63407x^7 + 0.36593x^8$$

$\widehat{\delta} = 0.4985$, $\delta/\widehat{\delta} = 0.9941$

Other Applications

Differential Evolution is oblivious to the nature of the objective function.

Can be used to design very good low-density parity-check codes over nontrivial channels (AWGN channel, BSC,...).

(Richardson-Shokrollahi-Urbanke.)

Conclusion

1. Design of erasure codes requires solving highly non-linear optimization problem.

2. A relaxed version can be solved via linear programming.

3. Linear programming solutions require codes to have huge lengths.

4. Differential evolution can be used to solve the non-linear problem. Solutions are excellent.