Capacity Achieving Codes for the AWGN Channel

Raj and Hesam

I. Basic Setup

We consider the setup shown in Fig. 1. A vector $\mathbf{x} \in \mathbb{F}_2^k$ is mapped into a codeword $\mathbf{y} \in \mathbb{F}_2^n$ by an

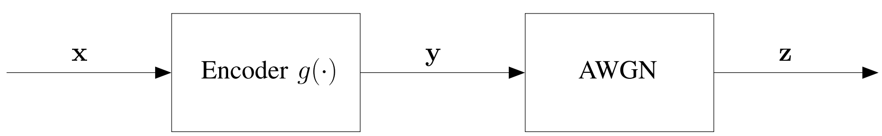

Fig. 1. Transmission over an AWGN

encoding function $g(\cdot)$. Since we will only consider linear codes, the encoder $g(\cdot)$ is a linear transformation from $\mathbb{F}_2^k$ to $\mathbb{F}_2^n$, with code generator matrix $\mathbf{G} \in \mathbb{F}_2^{k \times n}$. Hence

$$\mathbf{y}^{\mathsf{T}} = \mathbf{x}^{\mathsf{T}} \mathbf{G}.$$

The codeword $\mathbf{y}$ is sent over a (real) AWGN channel[1], whose output $\mathbf{z} \in \mathbb{R}^n$ may be written as

$$\mathbf{z} = \mathbf{y} + \mathbf{w},$$

where $\mathbf{w} \sim$ i.i.d. $\mathcal{N}\left(0, \frac{1}{\rho}\mathbf{I}\right)$. Hence ρ denotes the SNR of the AWGN channel.

Lemma 1: For the setup under consideration, $H(\mathbf{z}|\mathbf{x}) = H(\mathbf{z}|\mathbf{y})$, and hence $I(\mathbf{x};\mathbf{z}) = I(\mathbf{y};\mathbf{z})$.

[1]We assume standard BPSK modulation for transmission over the AWGN channel, i.e., $y_j \mapsto (-1)^{y_j}\ \forall\ j$. With a slight abuse of notation, we refer to both the binary codeword and the modulated symbols with the same notation $\mathbf{y}$; the one being referred to will be clear from the context.

Proof: We assume that the decoder is aware of the generator matrix $\mathbf{G}$. Intuitively, Lemma 1 is true since $\mathbf{y}$ is a deterministic function of $\mathbf{x}$. More rigorously, denote $N(\mathbf{G})$ as the cardinality of the left null-space of $\mathbf{G}$. We first notice that

$$p_{X|Z}(\mathbf{x}|\mathbf{z}) = \frac{1}{N(\mathbf{G})} p_{Y|Z}(\mathbf{G}^{\mathsf{T}}\mathbf{x}|\mathbf{z}) = \frac{1}{N(\mathbf{G})} p_{Y|Z}(\mathbf{y}|\mathbf{z}).$$

Further, we assume that the input is equiprobable, i.e., $p_X(\mathbf{x}) = \frac{1}{2^k}\ \forall\ \mathbf{x}$. Hence,

$$\begin{aligned} H(\mathbf{z}|\mathbf{x}) &= -\int\int p_{X,Z}(\mathbf{x},\mathbf{z}) \log p_{Z|X}(\mathbf{z}|\mathbf{x})\ d\mathbf{z}d\mathbf{x} \\ &= -\int_{\mathbf{z}} \sum_{\mathbf{x}} p_Z(\mathbf{z}) p_{X|Z}(\mathbf{x}|\mathbf{z}) \log \frac{p_Z(\mathbf{z}) p_{X|Z}(\mathbf{x}|\mathbf{z})}{p_X(\mathbf{x})}\ d\mathbf{z} \\ &= -\int_{\mathbf{z}} \sum_{\mathbf{x}} p_Z(\mathbf{z}) \frac{p_{Y|Z}(\mathbf{G}^{\mathsf{T}}\mathbf{x}|\mathbf{z})}{N(\mathbf{G})} \log \frac{p_Z(\mathbf{z}) p_{Y|Z}(\mathbf{G}^{\mathsf{T}}\mathbf{x}|\mathbf{z})}{N(\mathbf{G})\frac{1}{2^k}}\ d\mathbf{z}. \end{aligned}$$

In the summation above, as $\mathbf{x}$ runs over $\mathbb{F}_2^k$, $\mathbf{y}$ runs over the code $\mathcal{C}$, with each value being taken $N(\mathbf{G})$ times. Therefore,

$$\begin{aligned} H(\mathbf{z}|\mathbf{x}) &= -\int_{\mathbf{z}} N(\mathbf{G}) \sum_{\mathbf{y}\in\mathcal{C}} p_Z(\mathbf{z}) \frac{p_{Y|Z}(\mathbf{y}|\mathbf{z})}{N(\mathbf{G})} \log \frac{p_Z(\mathbf{z}) p_{Y|Z}(\mathbf{y}|\mathbf{z})}{p_Y(\mathbf{y})}\ d\mathbf{z} \\ &= -\int_{\mathbf{z}} \sum_{\mathbf{y}\in\mathcal{C}} p_{Y|Z}(\mathbf{y},\mathbf{z}) \log p_{Z|Y}(\mathbf{z}|\mathbf{y})\ d\mathbf{z} \\ &= H(\mathbf{z}|\mathbf{y}). \end{aligned}$$

■

In order to obtain lower bounds on $I(\mathbf{x};\mathbf{z})$, we will therefore analyse $I(\mathbf{y};\mathbf{z})$. Notice that

$$\begin{aligned} I(\mathbf{y};\mathbf{z}) &= H(\mathbf{z}) - H(\mathbf{z}|\mathbf{y}) \\ &= H(\mathbf{z}) - \frac{1}{2}\log\frac{(2\pi e)^n}{\rho^n}. \end{aligned} \tag{1}$$

II. Lower bound on the entropy

We now lower bound the entropy

$$H(\mathbf{z}) = -\int p(\mathbf{z}) \log p(\mathbf{z})\ d\mathbf{z}.$$

Using Jensen's inequality,

$$\log\left(\mathbb{E}_{f_1}(f_2(x))\right) \geq \mathbb{E}_{f_1}\left(\log f_2(x)\right).$$

Set $f_1(\cdot)$ and $f_1(\cdot)$ to equal $p(\mathbf{z})$, we obtain

$$H(\mathbf{z}) \geq -\log\left[\int p^2(\mathbf{z})\ d\mathbf{z}\right]. \tag{2}$$

Let the code $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{2^k}\}$. Notice that not all $\mathbf{c}_i$ are distinct, if $\mathbf{G}$ is not full-rank. The pdf of $\mathbf{z}$ is given by

$$p(\mathbf{z}) = \frac{1}{|\mathcal{C}|}\sum_{i=1}^{|\mathcal{C}|}\mathcal{N}\left(\mathbf{z}; \mathbf{c}_i, \frac{1}{\rho}\mathbf{I}\right) = \frac{1}{|\mathcal{C}|}\sum_{i=1}^{|\mathcal{C}|}\frac{\rho^{n/2}}{(\sqrt{2\pi})^n}e^{-\frac{\rho}{2}|\mathbf{z}-\mathbf{c}_i|^2}.$$

$$\int p^2(\mathbf{z})\ d\mathbf{z} = \frac{\rho^n}{|\mathcal{C}|^2(2\pi)^n}\int\sum_{i,j=1}^{|\mathcal{C}|}\exp\left\{-\frac{\rho}{2}\left[|\mathbf{z}-\mathbf{c}_i|^2 + |\mathbf{z}-\mathbf{c}_j|^2\right]\right\}\ d\mathbf{z}. \quad (3)$$

To evaluate the above integral, we make use of the following lemma.

Lemma 2: For $\mathbf{z}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, we have

$$|\mathbf{z}-\mathbf{a}|^2 + |\mathbf{z}-\mathbf{b}|^2 = 2\left|\mathbf{z} - \frac{\mathbf{a}+\mathbf{b}}{2}\right|^2 + \frac{|\mathbf{a}-\mathbf{b}|^2}{2}.$$

Proof: The left hand side may be written as

$$|\mathbf{z}-\mathbf{a}|^2 + |\mathbf{z}-\mathbf{b}|^2 = 2|\mathbf{z}|^2 + |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{z}^\mathsf{T}(\mathbf{a}+\mathbf{b}).$$

The right hand side evaluates to

$$2\left|\mathbf{z} - \frac{\mathbf{a}+\mathbf{b}}{2}\right|^2 + \frac{|\mathbf{a}-\mathbf{b}|^2}{2} = 2|\mathbf{z}|^2 + \frac{|\mathbf{a}+\mathbf{b}|^2}{2} - 2\mathbf{z}^\mathsf{T}(\mathbf{a}+\mathbf{b}) + \frac{|\mathbf{a}-\mathbf{b}|^2}{2} = 2|\mathbf{z}|^2 + |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{z}^\mathsf{T}(\mathbf{a}+\mathbf{b}).$$

■

Using the results of Lemma 2 in (3), we obtain

$$\begin{aligned}\int p^2(\mathbf{z})\ d\mathbf{z} &= \frac{\rho^n}{(2\pi)^n|\mathcal{C}|^2}\sum_{i,j=1}^{|\mathcal{C}|}\int\exp\left\{-\frac{\rho}{2}\left[2\left|\mathbf{z}-\frac{\mathbf{c}_i+\mathbf{c}_j}{2}\right|^2 + \frac{|\mathbf{c}_i-\mathbf{c}_j|^2}{2}\right]\right\}d\mathbf{z} \\ &= \frac{\rho^n}{(2\pi)^n|\mathcal{C}|^2}\sum_{i,j=1}^{|\mathcal{C}|}\exp\left\{-\frac{\rho}{4}|\mathbf{c}_i-\mathbf{c}_j|^2\right\}\int\exp\left\{-\rho\left|\mathbf{z}-\frac{\mathbf{c}_i+\mathbf{c}_j}{2}\right|^2\right\}d\mathbf{z} \\ &= \frac{\rho^n}{(2\pi)^n|\mathcal{C}|^2}\sqrt{\frac{(2\pi)^n}{(2\rho)^n}}\sum_{i,j=1}^{|\mathcal{C}|}\exp\left\{-\frac{\rho}{4}|\mathbf{c}_i-\mathbf{c}_j|^2\right\} \quad (4)\end{aligned}$$

From (1), (2) and (4), we obtain

$$I(\mathbf{x};\mathbf{z}) \geq -\log\left[\left(\frac{e}{2}\right)^{n/2}\frac{1}{|\mathcal{C}|^2}\sum_{i,j=1}^{|\mathcal{C}|}\exp\left\{-\frac{\rho}{4}|\mathbf{c}_j-\mathbf{c}_i|^2\right\}\right].$$

Since BPSK modulation is used, $|\mathbf{c}_j - \mathbf{c}_i|^2 = 4d_H(\mathbf{c}_j, \mathbf{c}_i)$. Further, since we employ a linear code, we may further simplify the above to obtain

$$I(\mathbf{x};\mathbf{z}) \geq -\log\left[\left(\frac{e}{2}\right)^{n/2} \frac{1}{|\mathcal{C}|} \sum_{i=1}^{|\mathcal{C}|} \exp\left\{-\rho\ w_H(\mathbf{c}_i)\right\}\right],$$

where $w_H(\mathbf{x})$ denotes the Hamming weight of $\mathbf{x}$. If we define B_w to be the number of codewords with Hamming weight w, we may rewrite the above as:

$$I(\mathbf{x};\mathbf{z}) \geq -\log\left[\left(\frac{e}{2}\right)^{n/2} \frac{1}{|\mathcal{C}|} \sum_{w=0}^{n} B_w e^{-w\rho}\right] \tag{5}$$

We now use (5) to bound the probability of a code achieving capacity over the AWGN channel, as follows. Let $\epsilon > 0$ denote a constant, and $\text{Cap}(C)$ denote the capacity of the AWGN channel constrained to using binary inputs.

$$\begin{aligned} P\{I(\mathbf{x};\mathbf{z}) < n(\text{Cap}(C)-\epsilon)\} &\leq P\left\{-\log\left[\left(\frac{e}{2}\right)^{n/2} \frac{1}{|\mathcal{C}|} \sum_{w=0}^{n} B_w e^{-w\rho}\right] < n(\text{Cap}(C)-\epsilon)\right\} \\ &= P\left\{\left(\sqrt{\frac{e}{2}} 2^{\text{Cap}(C)-R}\right)^n \sum_{w=0}^{n} B_w e^{-w\rho} > 2^{n\epsilon}\right\}. \end{aligned} \tag{6}$$

In order to make statements regarding the above probability, we present in the following section an analysis of the quantity

$$\mathcal{S} = b^n \sum_{w=1}^{n} B_w e^{-\rho w},$$

where b is a constant that is independent of n.

III. Distance Spectrum of the Code

In order to analyze B_w, we consider two scenarios. When $n \geq k$, we use a good channel code to transmit information across the channel. On the other hand, when $n < k$, we need to compress (quantize) the information to be sent over the channel. We first examine the case when $n \geq k$.

A. Channel coding when $n \geq k$

In order to define our channel coding ensemble, consider the following linear equation in GF(2):

$$e_{i_1} + \cdots + e_{i_d} = 0, \tag{7}$$

where $i_1, \ldots, i_d$ are i.i.d. random variables taking on the values $\{1, 2, \ldots, n-k\}$ with equal probability. If an even number of these random variables take on the same value j, then e_j does not appear in the equation, owing to the mod-two addition over GF(2). We may associate with the equation in (7) a column vector $\mathbf{e}$, defined as follows. The j^{th} coordinate of $\mathbf{e}$ is equal to one if e_j appears (i.e., does not cancel out) in (7), else it is equal to zero.

We define our channel coding LDPC ensemble in the following way. Generate the parity check matrix $\mathbf{H} \in \text{GF}(2)^{(n-k)\times n}$ such that each column of $\mathbf{H}$ is drawn independently from the column vector ensemble defined above. Hence each column of $\mathbf{H}$ can have a maximum of d ones. The quantity B_w is equal to the number of vectors with weight w in the right null-space of $\mathbf{H}$, or in the left null-space of $\mathbf{H}^{\mathsf{T}}$.

Along the lines of the proof of Theorem 3.5.1 in Sec. 3.6 of [1], we analyze $\mathcal{S}$ by splitting the sum into three regions,

$$\mathcal{S}_1 = b^n \sum_{1\le w<\delta n} B_w e^{-\rho w},$$
$$\mathcal{S}_2 = b^n \sum_{\delta n\le w<(1-\delta)n} B_w e^{-\rho w},$$
$$\mathcal{S}_3 = b^n \sum_{(1-\delta)n\le w\le n} B_w e^{-\rho w},$$

where $\delta \to 0$.

1) Analysis of $\mathcal{S}_1$*:* From the proof of Lemma 3.5.1 in [1], we have that

$$\begin{aligned}\mathcal{S}_1 &= b^n \sum_{1\le w<\delta n} B_w e^{-\rho w} \\ &\le b^n \sum_{1\le w<\delta n} \left[\left(\frac{1}{1-R}\right)^{d/2} d^{\frac{d}{2}-1} e^{4d} \delta^{\frac{d}{2}-1} e^{-\rho}\right]^n \\ &\le \epsilon,\end{aligned}$$

for some $\delta > 0$, where $\epsilon > 0$ is a constant.

2) Analysis of $\mathcal{S}_3$*:* Define $\lambda = \frac{dw}{n-k}$. We make use of the proof of Lemma 3.5.2 in [1] to show that

$$\begin{aligned}\mathcal{S}_3 &= b^n \sum_{(1-\delta)n\le w\le n} B_w e^{-\rho w} \\ &\le \frac{c(dn)^{1/2} e^{\rho w_0}}{(1-q)^{w_0}} \left[b\ e^{-\rho - R\ln q}\right]^n,\end{aligned}$$

where w_0 is an integer such that $w_0 \le n\delta$, and $q < 1$ is such that

$$q \ge \frac{1+e^{-2\lambda}}{2}. \tag{8}$$

For $\mathcal{S}_3$ to vanish, it is sufficient that

$$\ln b - \rho - R\ln q < 0.$$

Using (8), it is sufficient to ensure that

$$\ln b - \rho - R \ln\left(\frac{1+e^{-2\lambda}}{2}\right) < 0. \tag{9}$$

Notice that in the range of summation of Sc_3,

$$\frac{d(n-w_0)}{n-k} \leq \lambda \leq \frac{dn}{n-k}.$$

As $n, k \to \infty$, we hence have that

$$\lambda \to \frac{dn}{n-k} = \frac{d}{1-R}.$$

We thus simplify (9) to obtain

$$\frac{b2^R}{\left(1+e^{-2\frac{d}{1-R}}\right)^R} < e^{\rho}$$

In Fig. 2, we plot the behaviour of the function $f(R) = \left(1+e^{-2\frac{d}{1-R}}\right)^R$, for an illustrative example of $d = 5$. It is seen that this function is sharply concentrated around 1, for all $0 \leq R \leq 1$. We may refine

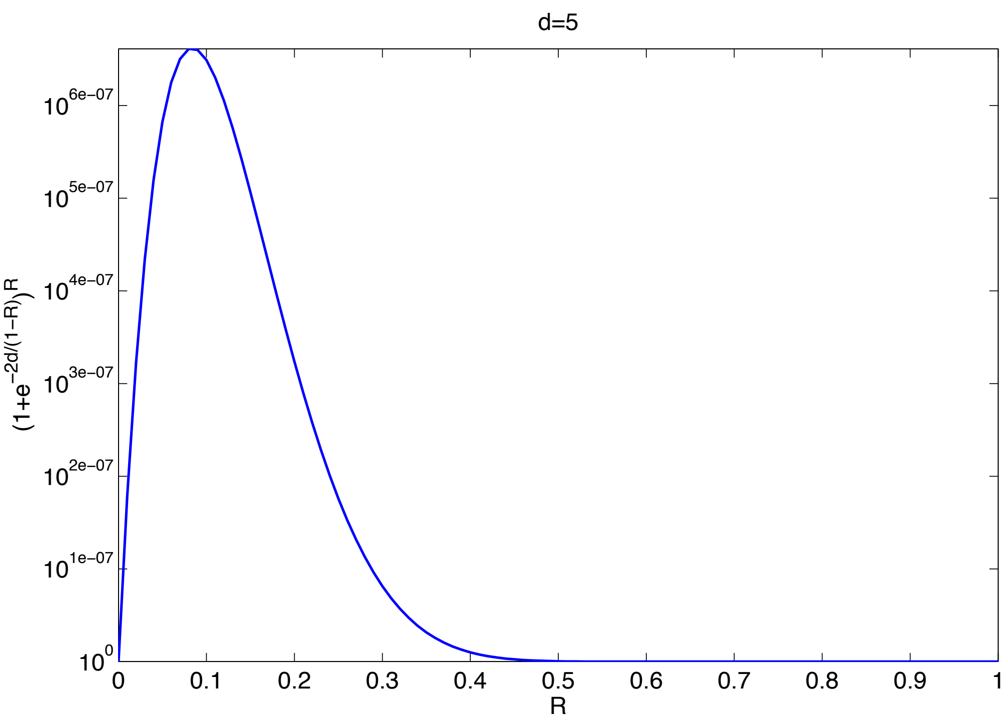

Fig. 2. Behaviour of the function $f(R) = \left(1+e^{-2\frac{d}{1-R}}\right)^R$

our sufficient condition to

$$b2^R e^{-\rho} < 1.$$

Substituting the value of $b = \sqrt{\frac{e}{2}} 2^{\text{Cap}(C)-R}$, we obtain

$$\sqrt{\frac{e}{2}} 2^{\text{Cap}(C)} e^{-\rho} < 1.$$

Since $\text{Cap}(C) < \frac{1}{2}\log_2(1+\rho)$, it is sufficient that

$$\sqrt{\frac{e}{2}} (1+\rho)^{\frac{1}{2}} e^{-\rho} < 1.$$

Shown in Fig. 3 is a plot of the function $f(\rho) = e^{-\rho}(1+\rho)^{1/2}$. We hence observe that if we ignore the

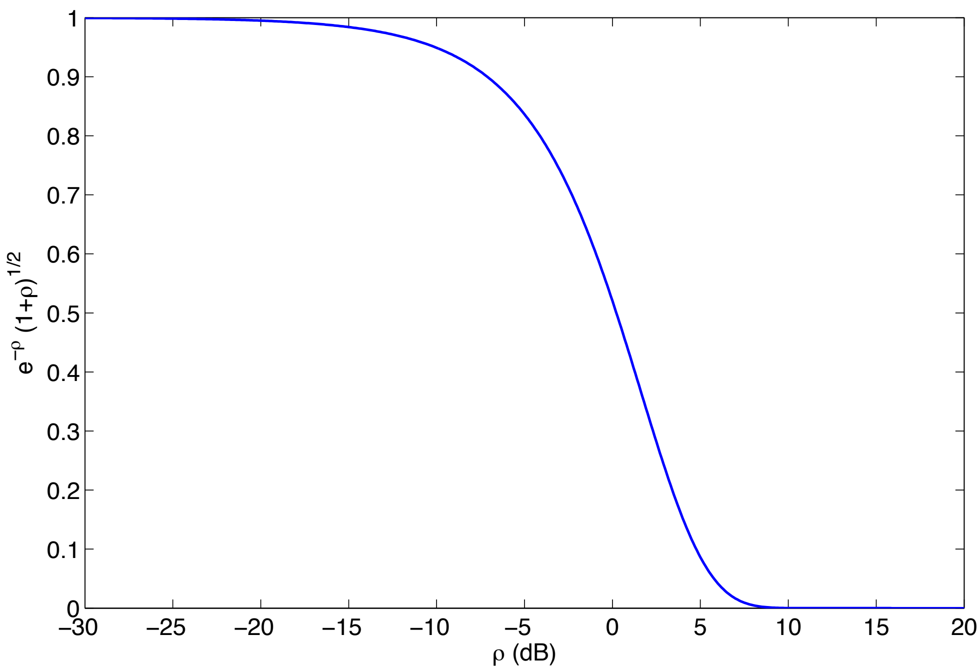

Fig. 3. Behaviour of the function $f(\rho) = e^{-\rho}(1+\rho)^{1/2}$

factor of $\sqrt{\frac{e}{2}}$ in the expression for b, $\mathcal{S}_3$ converges to zero for all ρ, R. This factor may be accounted for by backing off from capacity by $\epsilon = \log_2\left(\sqrt{\frac{e}{2}}\right)$ bits in (6).

3) Analysis of $\mathcal{S}_2$: By replacing 'k' with '$n-k$' in equation (7) in [2], we obtain that

$$b^n B_w e^{-\rho w} \leq \left(b \left(\frac{ne}{w}\right)^{\frac{w}{n}} (\cosh \lambda)^{\frac{n-k}{n}} \left(\frac{wd}{\lambda(n-k)e}\right)^{\frac{wd}{n}} e^{-\frac{\rho w}{n}} \sqrt[n]{wd} \right)^n .$$

We not make a change of variables $x = \frac{w}{n-k}$ and $\phi = \frac{n}{n-k}$. Hence $\mathcal{S}_2$ vanishes if

$$\phi \ln b + x \ln\left(\frac{\phi e}{x}\right) + \ln\cosh\lambda + xd\ln\left(\frac{xd}{\lambda e}\right) - \rho x < 0.$$

Since λ is arbitrary, we choose $x = \frac{\lambda\tanh\lambda}{d}$ as in [1], [2], to simplify the above condition to

$$f(\lambda, \phi) < 0, \tag{10}$$

where

$$f(\lambda,\phi) = \ln\left[b^{\phi}\left(\frac{ed\phi}{\lambda\tanh\lambda}\right)^{\frac{\lambda\tanh\lambda}{d}}\right] - \frac{\rho\lambda\tanh\lambda}{d} + \ln\cosh\lambda + \lambda\tanh\lambda\ln\left(\frac{\tanh\lambda}{e}\right),$$

and $\phi \to \frac{1}{1-R}$ as $n, k \to \infty$.

We may use the alternative expression for the factorial function to obtain an alternative bound on the region where $\mathcal{S}_2$ vanishes, as in [1], [2]. From the corresponding expression in [2], we obtain that

$$b^n B_w e^{-\rho w} \le \frac{2\phi b^n}{\sqrt{2\pi x(\phi - x)\phi(n-k)}}\left(\frac{\phi^{\phi}(\phi-x)^x}{x^x(\phi-x)^{\phi}}\right)^{n-k}\cosh(\lambda)^{n-k}\left(\frac{wd}{\lambda e(n-k)}\right)^{wd} wd\, e^{-\rho w}(1+o(1)),$$

where $x = \frac{w}{n-k}$ and $\phi = \frac{n}{n-k}$. We set $x = \frac{\lambda\tanh\lambda}{d}$ and simplify the above to

$$b^n B_w e^{-\rho w} \le \frac{2\phi wd}{\sqrt{2\pi x(\phi-x)\phi(n-k)}} \cdot \left[\frac{(\phi d)^{\phi}(d\phi - \lambda\tanh\lambda)^{\frac{\lambda\tanh\lambda}{d}-\phi}\cosh\lambda}{(\lambda\tanh\lambda)^{\frac{\lambda\tanh\lambda}{d}}}\left(\frac{\tanh\lambda}{e}\right)^{\lambda\tanh\lambda} e^{-\frac{\rho}{d}\lambda\tanh\lambda}\; b^{\frac{n}{n-k}}\right]^{n-k}(1+o(1)).$$

Note that $x = \frac{w}{n-k} \le \frac{n}{n-k} = \phi$. In order for $\mathcal{S}_2$ to vanish, it is sufficient that

$$\frac{(\phi d)^{\phi}(d\phi - \lambda\tanh\lambda)^{\frac{\lambda\tanh\lambda}{d}-\phi}\cosh\lambda}{(\lambda\tanh\lambda)^{\frac{\lambda\tanh\lambda}{d}}}\left(\frac{\tanh\lambda}{e}\right)^{\lambda\tanh\lambda} e^{-\frac{\rho}{d}\lambda\tanh\lambda}\; b^{\phi} < 1$$

$$\Leftrightarrow \rho > g(\lambda, \phi), \tag{11}$$

where

$$g(\lambda,\phi) \triangleq \frac{d}{\lambda\tanh\lambda}\ln\left[\frac{(\phi d)^{\phi}(d\phi - \lambda\tanh\lambda)^{\frac{\lambda\tanh\lambda}{d}-\phi}\cosh\lambda}{(\lambda\tanh\lambda)^{\frac{\lambda\tanh\lambda}{d}}}\left(\frac{\tanh\lambda}{e}\right)^{\lambda\tanh\lambda} b^{\phi}\right].$$

Note that $\phi \to \frac{1}{1-R}$ as $n, k \to \infty$.

Hence for all pairs (R, ϕ) such that either (10) or (11) holds, we are guaranteed that $\mathcal{S}$ vanishes, and that we are within $\epsilon = \log_2\left(\sqrt{\frac{e}{2}}\right)$ bits of the capacity of the channel.

B. *Compression (Quantization) for* $n < k$

When $n < k$, we move from the realm of channel coding to the case where we have to compress information. We continue to focus on the case of a linear "encoding" function $\mathbf{G} \in \mathbb{F}_2^{k \times n}$, that compresses $\mathbf{x} \in \mathbb{F}_2^k$ to $\mathbf{y} \in \mathbb{F}_2^n$ through the map $\mathbf{y}^\mathsf{T} = \mathbf{x}^\mathsf{T}\mathbf{G}$. Intuitively, in order to maximize the mutual information between $\mathbf{z}$ and $\mathbf{x}$, we would like to choose $\mathbf{G}$ to be of full-rank n. If $\mathbf{G}$ is of rank n, then as $\mathbf{x}$ varies over all k-tuples, $\mathbf{y}$ varies over all n-tuples, with each n-tuple appearing 2^{k-n} times in the quantizer codebook. Hence irrespective of the particular matrix $\mathbf{G}$ that we choose, as long as $\mathbf{G}$ is full-rank, we have that

$$B_w = 2^{k-n}\binom{n}{w}.$$

Consider

$$\begin{aligned} b^n \sum_{w=0}^{n} B_w e^{-\rho w} &= b^n 2^{k-n} \sum_{w=0}^{n} \binom{n}{w} e^{-\rho w} \\ &= b^n 2^{k-n} \left(1 + e^{-\rho}\right)^n . \end{aligned}$$

We may now evaluate (6) as

$$P\{I(\mathbf{x};\mathbf{z}) < n(\text{Cap}(C) - \epsilon)\} \le P\left\{ b 2^{\frac{k}{n}-1}(1 + e^{-\rho}) > 2^\epsilon \right\}.$$

Suppose that we back-off from capacity by setting $\epsilon = \log_2\left(\sqrt{\frac{e}{2}}\right)$, as before. Then the above reduces to

$$P\{I(\mathbf{x};\mathbf{z}) < n(\text{Cap}(C) - \epsilon)\} \quad \le \quad P\left\{ 2^{\text{Cap}(C)} \left(\frac{1 + e^{-\rho}}{2} \right) > 1 \right\}.$$

Since $\text{Cap}(C)$ is not known in closed form, we numerically compute the function $2^{\text{Cap}(C)}\left(\frac{1+e^{-\rho}}{2}\right)$ and plot it in Fig. 4. From this plot, it is evident that once we increase the rate $R > 1$, we are guaranteed to be within ϵ bits from capacity.

IV. SIMULATION RESULTS FOR THE THRESHOLD

In this section, we examine the threshold behaviour of the quantity $\mathcal{S}$, for rates $R \le 1$ through simulations. For fixed ρ, we examine the minimum rate above which $\mathcal{S}$ converges to zero (i.e., the minimum rate for which either (10) or (11) holds). Some initial simulation results of the threshold rate at $d = 5$ are shown in the Table I. For comparison, we plot in Fig. 5 the capacities of the unconstrained AWGN channel, and the binary constrained AWGN channel. Notice that the values of the threshold that we obtain are almost identical to the binary constrained capacity of the AWGN channel.

In conclusion, apart from the $\epsilon = \log_2\left(\sqrt{\frac{e}{2}}\right)$ bits that we have to backoff owing to looseness in our bounds, the threshold seems to point strongly to the fact that the ensemble of codes that we consider are capacity achieving.

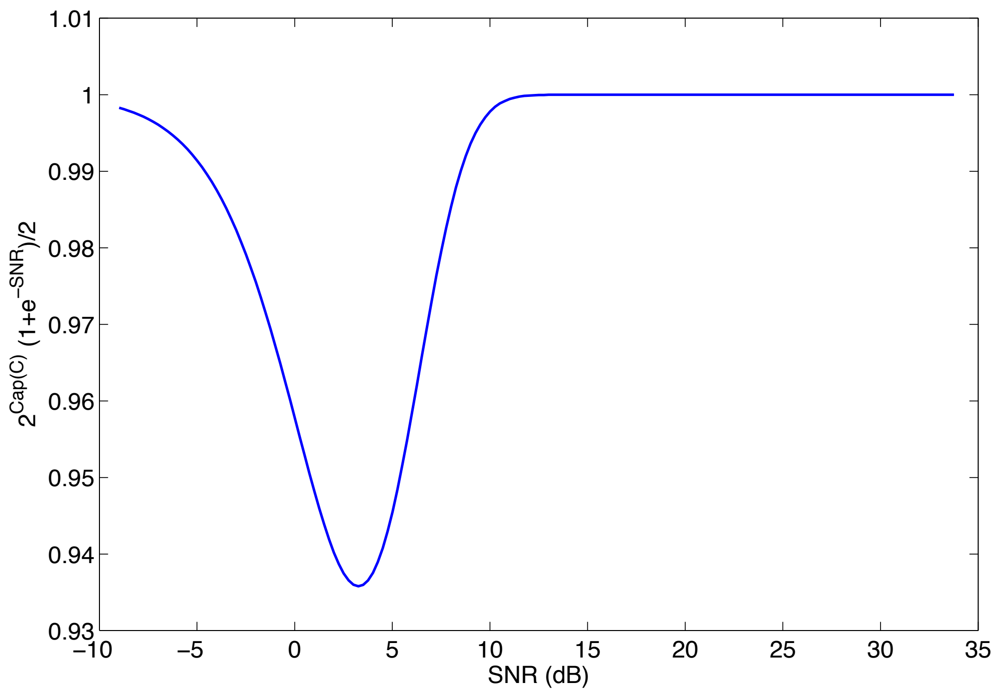

Fig. 4. Behaviour of the function $2^{\text{Cap}(C)}\left(\frac{1+e^{-\rho}}{2}\right)$

TABLE I

THRESHOLD FOR THE MUTUAL INFORMATION OVER THE AWGN CHANNEL

SNR ρ (dB)	Threshold (bits)
-5	0.2
0	0.5
5	0.9

REFERENCES

[1] V. F. Kolchin, *Random Graphs*, Encyclopedia of Mathematics, Vol. 53, Cambridge University Press, 1999.

[2] P. Pakzad and A. Shokrollahi, "Phase Transitions for Mutual Information," Draft manuscript, May 2009.

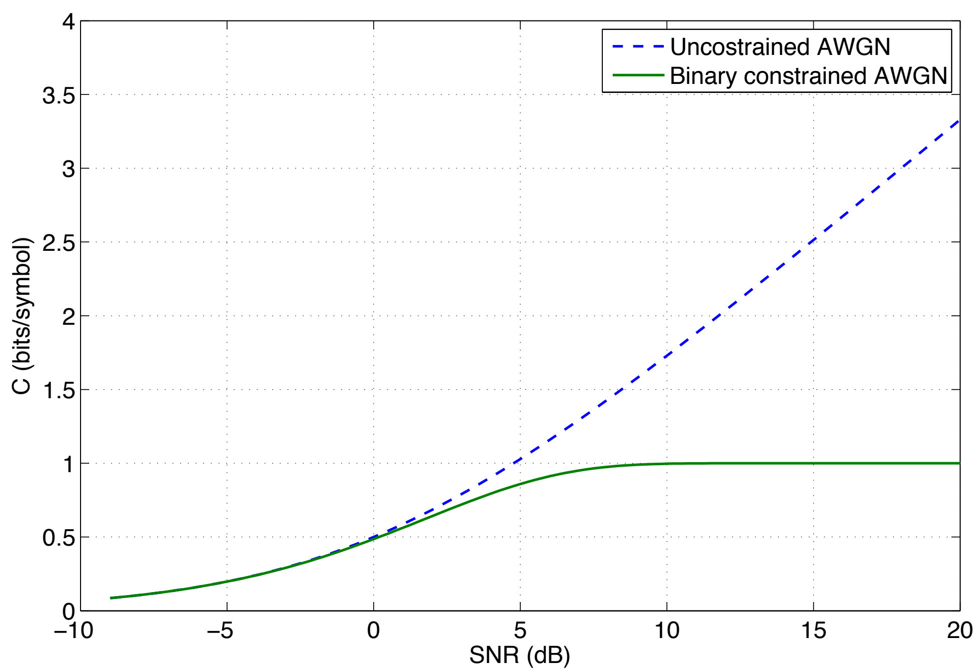

Fig. 5. Capacity of the unconstrained and binary constrained AWGN channels