We need to work with a distribution D over $\mathbb{F}_{2N}^n$

Suppose that $2N = 2^m$, some m. Then we equivalently work on a distribution D' over $\mathbb{F}_2^{mn}$

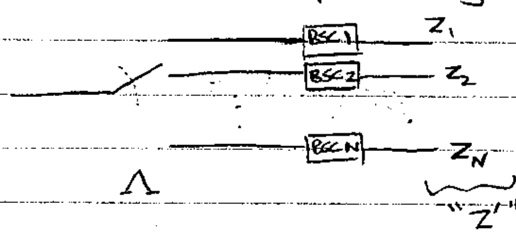
<u>Claim</u>: $H(D) = H(D')$ (check)

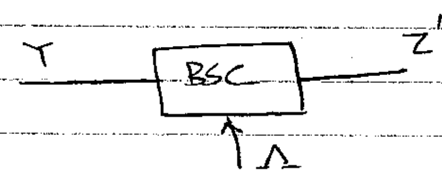
13/04/10 <u>Equivalence of channels</u>:

From the CBSC entropy computation, the o/p entropy of the original CBSC is

$$H(Z) = H(\underline{\lambda}) + \sum_{i=1}^{N} \lambda_i H(Z_i)$$

Consider the following channel:





$$H(\Lambda, Z_1, Z_2, \ldots, Z_N) = H(\Lambda) + H(Z_1|\Lambda) + H(Z_2|Z_1,\Lambda) + \cdots$$

$$= H(\Lambda) + \lambda_1 H(Z_1 | \Lambda \text{ in switch posn } 1) + \lambda_2 H(Z_2 | \Lambda \text{ in switch posn } 2) + \cdots$$

$$= H(\underline{\lambda}) + \underbrace{\sum_{i=1}^{N} \lambda_i H(Z_i)}_{= H(Z'|\Lambda)}$$

Hence the channel

$Y \to Z$ (original CBSC)

is equivalent to the channel

$Y \to (\Lambda, Z')$ in terms of output entr

To compute $H(Z'|\Lambda)$, we replace "p" in the BSC proof with $\sum_{i=1}^{N} \lambda_i p_i$

$$H(Z) = H(\Lambda, Z') = H(Z') + H(\Lambda / Z') \geq H(Z')$$

Since Z' is the o/p of a "compound-BSC",

$$p(Z') = \sum_{i=1}^{N} \lambda_i P_i(Z) \quad \text{(all as n-dim vectors)}$$

This induces a distribution on $\mathbb{F}_2^n$, and we may use the BSC theorem here, with transition probability $\sum_{i=1}^{N} p_i \lambda_i$.

$$\sum_{i=1}^{N} \lambda_i H(Z_i) = -\sum_{i=1}^{N} \lambda_i \sum_y \sum_z p(y) P_i(z|y) \log P_i(z|y)$$

$$= -\sum_y p(y) \sum_z \sum_{i=1}^{N} \lambda_i P_i(z/y) \log P_i(z/y)$$

$$\geq \cancel{-\sum_y p(y) \sum_z \sum_{i=1}^{N} \lambda_i P_i(z/y) \log \sum_{j=1}}$$

$$\Pr\{ I(x;z) < n\,\mathrm{Cap}(\mathcal{C}) \} \leq n[1 + H(\lambda)] - \mathbb{E}[H(Z')]$$

$$\leq n H(\lambda) + \log_2\left(\sum_{w=0}^{n} B_w (1-2p)^{2w} \right)$$

(Not a useful bound)

pdf of the o/p of the compound-BSC:

$$P(Z) = \sum_{i=1}^{N} \lambda_i P_i(Z) \qquad P_i(\cdot) \to \text{pdf of o/p of BSC}$$

$$H(Z) = -\sum_Z p(Z) \log p(Z)$$

$$= -\sum_Z \sum_{i=1}^{N} \lambda_i P_i(Z) \log \sum_{j=1}^{N} \lambda_j \log P_j(Z)$$

$N=3$

$n=5$

$$H(Z) = H(\underline{\lambda}) + \sum_{i=1}^{N} \lambda_i H(Z^{(i)})$$

o/p of BSC(p_i)

$$H(Z_1) = H(\underline{\lambda}) + \sum_{i=1}^{N} \lambda_i H(Z_1^{(i)})$$

We know that $H(Z) = H(\Lambda, Z')$. Assume for the mo
that $H(Z^n) = H(\Lambda^n, Z'^n)$ as well.

$$H(\underline{\Lambda}^n; Z'^n) = H(\underline{\Lambda}^n) + H(Z'^n / \Lambda^n)$$

$$= n H(\underline{\lambda}) + \sum_{\forall s^n} P(s^n)\, H(Z'^n / \underline{\Lambda}^n = s^n)$$

$s_i \in \{1, \dots, N\}$

$$\geq n H(\underline{\lambda}) + \sum_{s^n} \left(\prod_{i=1}^{n} \lambda_{s_i}\right) \left[n - \log_2 \left(\sum_{\underline{c} \in C^\perp} \prod_{c_i = 1}^{n} (1-2p_{s_i}\right.\right.$$

$$= n H(\underline{\lambda}) + n \underbrace{\sum_{\forall s^n}}_{N^n \text{ terms}} \left(\prod_{i=1}^{n} \lambda_{s_i}\right)$$

$$- \sum_{s^n} \left(\prod_{i=1}^{n} \lambda_{s_i}\right) \log_2 \left(\sum_{\underline{c} \in C^\perp} \prod_{c_i=1}^{n} (1-2p_{s_i})^2 \right)$$

$$\geq n H(\underline{\lambda}) + n \sum_{\forall s^n} \left(\prod_{i=1}^{n} \lambda_{s_i}\right)$$

$$- \sum_{s^n} \left(\prod_{i=1}^{n} \lambda_{s_i}\right) \log_2 \left(\sum_{w=0}^{n} B_w (1-2p\right.$$

where $p = \min_i p_i$

$\therefore \Pr\{ I(X;Z) < n\, Cap(\mathcal{C}) \}$

$$\leq \cancel{n - n \underbrace{\sum_{\forall s^n} \left(\prod_{i=1}^{n} \lambda_{s_i}\right)}_{= \left(\sum_{i=1}^{N} \lambda_i\right)^n = 1}} + \sum_{s^n} \left(\prod_{i=1}^{n} \lambda_{s_i}\right) \log_2 \left(\sum_{w=0}^{n} B_w\right.$$

$$= \log_2 \left(\sum_{w=0}^{n} B_w (1-2p)^{2w} \right)$$

$$\Pr\{I(X;Z) < n\,\mathrm{Cap}(\mathcal{C})\} \le \underbrace{\sum_{s^n \in [N]^n} \left(\prod_{i=1}^{n} \lambda_{s_i}\right) \log_2 \Big(\underbrace{\sum_{\underline{c} \in C^\perp} \prod_{c_i = 1} (1-2p_{s_i})^2}_{\triangleq f(s^n)} \Big)}_{\mathbb{E}[\log f(s^n)]}$$

$$\le \log \mathbb{E}\{f(s^n)\}$$

$$\le \log_2 \left[\sum_{s^n \in [N]^n} \left(\prod_{i=1}^{n} \lambda_{s_i}\right) \sum_{\underline{c} \in C^\perp} \prod_{c_i=1} (1-2p_{s_i})^2 \right]$$

$$= \log_2 \Bigg[\sum_{\underline{c} \in C^\perp} \underbrace{\sum_{s^n \in [N]^n} \left(\prod_{i=1}^{n} \lambda_{s_i}\right) \prod_{c_i=1} (1-2p_{s_i})^2}_{\text{fixed } \underline{c}\text{: will have } w_H(\underline{c}) \text{ prod. terms}} \Bigg]$$

$$\le \log_2 \left[\sum_{\underline{c} \in C^\perp} \sum_{s^n \in [N]^n} \prod_{c_i=1} \lambda_{s_i} (1-2p_{s_i})^2 \right]$$

$$= \log_2 \Bigg[\sum_{\underline{c} \in C^\perp} \underbrace{\left[\sum_{i=1}^{N} \lambda_{s_i} (1-2p_{s_i})^2 \right]}_{\rho'}^{w_H(\underline{c})} \cdot N^{n - w_H(\underline{c})} \Bigg]$$

$$= \log_2 \left[\sum_{w=0}^{n} B_w \cdot N^{n-w} \rho'^{w} \right]$$

$$= \log_2 \left[N^n \sum_{w=0}^{n} B_w \rho^w \right] \quad \text{where } \rho = \frac{1}{N} \sum_{i=1}^{N} \lambda_i (1-2p_{s}$$

Now we follow the BSC proof:

Analyze $\quad b^n \sum_{w=1}^{n} \mathbb{E}\{B_w\}\, p^w$

<u>1st part</u>:

$$b^n \sum_{1 \le w \le \delta n} \mathbb{E}\{B_w\}\, p^w$$

$$\le b^n \sum_{1 \le w \le \delta n} \left[\binom{n}{k}^{d/2} d^{d/2 - 1} e^{4d} \delta^{d/2 - 1} p \right]^w \le \varepsilon \text{ for}$$

δ small en

$p < \frac{1}{2}$

$(1-2p)^2 < \frac{1}{2}$

$1-2p < \frac{1}{\sqrt{2}}$

$2p > 1 - \frac{1}{\sqrt{2}} \qquad p > \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$

$\alpha < 1 : \quad \frac{n}{k} < 1 \qquad n < k$

$b^T = \left(b^{\frac{T}{n-T}}\right)^{n-T}$

2^{nd} part: From 15 pages before, we will analyze (in Kolchin's notation)

$$b^T \sum_{m=T-m_0}^{T} \binom{T}{m} P_E(\lambda m, n)\; p^m$$

$$\leq b^T \cdot \frac{c\,(\lambda T)^{1/2}}{(1-q)^{m_0}} \left[\frac{p^{\left(1-\frac{m_0}{T}\right)}}{q^{(1-1/\alpha)}} \right]^T \qquad \delta = \frac{m_0}{T}$$

$$b\, p^{(1-\delta)} \overset{?}{<} q^{\left(1-\frac{1}{\alpha}\right)}$$

$$= \left(\frac{1+e^{-2\lambda}}{2}\right)^{\left(1-\frac{1}{\alpha}\right)}$$

$\lambda \to \alpha n$ as $T, n \to \infty$ (as before)

$$b\, p^{(1-\delta)} \overset{?}{<} \left(\frac{1+e^{-2\alpha n}}{2}\right)^{\left(1-\frac{1}{\alpha}\right)}$$

$$\Leftrightarrow \quad p < \left[\frac{1}{b}\left(\frac{1+e^{-2\alpha n}}{2}\right)^{\left(1-\frac{1}{\alpha}\right)}\right]^{\frac{1}{1-\delta}}$$

As $\delta \to 0$, we need

$$p < \underbrace{\frac{1}{b}\Bigg(\overbrace{\frac{1+e^{-2\alpha n}}{2}}^{\leq 1}\Bigg)^{\left(1-\frac{1}{\alpha}\right)}}_{\triangleq\, h(\alpha)}$$

$\alpha n \uparrow \quad \frac{1}{\alpha} \downarrow$

$\Rightarrow 1 - \frac{1}{\alpha} \uparrow$

Determine the max α s.t. $p < h(\alpha)$

As $\alpha \to \infty$, $h(\alpha) \to \frac{1}{2b}$ $\qquad h(\alpha)$ is a $\downarrow$ fn

If $p < \frac{1}{2b}$, then the middle part converges for all α (ie, all rates, however small)

3^{rd} part: From pg. 12 of Pakzad-Shokrollahi,

$$b^n \, \mathbb{E}[B_w] \, p^w \leq \left[b \left(\frac{ne}{w}\right)^{w/n} \cosh(\lambda)^{k/n} \left(\frac{wd}{\lambda k e}\right)^{wd/n} p^{w/n} \sqrt[n]{wd} \right]^n$$

$x = w/k$, $\phi = n/k$

$$\ln b + \frac{x}{\phi} \ln\left(\frac{\phi e}{x}\right) + \frac{1}{\phi} \ln\cosh(\lambda) + \frac{xd}{\phi} \ln\left(\frac{xd}{\lambda e}\right)$$
$$+ \frac{x}{\phi} \ln p < 0$$

$x = \dfrac{\lambda \tanh \lambda}{d}$:

$$\phi \ln b + \frac{\lambda \tanh\lambda}{d} \ln\left(\frac{\phi e d}{\lambda \tanh\lambda}\right) + \ln\cosh\lambda + \quad ⊛$$
$$\lambda \tanh\lambda \ln\left(\frac{\tanh\lambda}{e}\right) + \frac{\lambda \tanh\lambda}{d} \ln p < 0$$

$\lambda \tanh\lambda$ varies between $\frac{d}{k}\delta n$ and $(1-\delta)\frac{dn}{k}$

$\delta \to 0$: 0 and $d\phi$

$\therefore$ for all λ s.t. $\lambda\tanh\lambda < d\phi$, we need the LHS of ⊛ to be negative. Find the max ϕ s.t. this is true.

Approach 2, 3^{rd} part.

In $g(\lambda, \phi)$, replace "$(1-2p)^2$" with p, and add a term b^ϕ

NOTE ⊛ $\iff$ $$\frac{b^\phi \, (\phi e d)^{\frac{\lambda\tanh\lambda}{d}} \cosh\lambda \, (\tanh\lambda)^{\lambda\tanh\lambda}}{(\lambda\tanh\lambda)^{\frac{\lambda\tanh\lambda}{d}} \, e^{\lambda\tanh\lambda}} \cdot p^{\frac{\lambda\tanh\lambda}{d}} < 1$$

$$\Leftrightarrow \quad \frac{b^{\phi d/\lambda \tanh\lambda}}{\lambda \tanh\lambda} \cdot \frac{\phi e d\,(\cosh\lambda)^{d/\lambda\tanh\lambda}}{e^{d}} \quad (\tanh\lambda)^{d}\ p \quad <$$

TWO CHANNEL USES OF A CBSC:

$$P(Z^2 = 0_i 1_j) = (c_i\alpha + d_i)(e_j\alpha + f_j)$$
$$P(Z^2 = 1_i 0_j) = (e_i\alpha + f_i)(c_j\alpha + d_j)$$
$$P(Z^2 = 0_i 0_j) = (c_i\alpha + d_i)(c_j\alpha + d_j)$$
$$P(Z^2 = 1_i 1_j) = (e_i\alpha + f_i)(e_j\alpha + f_j) = \lambda_i\lambda_j \underbrace{(e_i'\alpha + f_i')}\underbrace{(e_j'\alpha +}$$

$$H(Z^2) = -\sum_{i,j=1}^{N} \lambda_i\lambda_j \Big[(c_i'\alpha + d_i')(e_j'\alpha + f_j') \log (c_i'\alpha + d_i')(e_j'\alpha + f_j') + (c_i'\alpha + d_i')(e_j'\alpha + f_j') \log \lambda_i\lambda_j + \cdots \Big]$$

$$= -\sum_{i,j=1}^{N} \lambda_i\lambda_j \big[\log(\lambda_i\lambda_j)\big]\,(1) \qquad +$$

Sum of o/p probs of 2 ch. uses of a BSC

$$\sum_{i,j=1}^{N} \lambda_i\lambda_j \underbrace{\Big[(c_i'\alpha + d_i')(e_j'\alpha + f_j') \log (c_i'\alpha + d_i')(e_j'\alpha + \cdots + \cdots \Big]}$$

is equal to:

$$H(Z'^2 \mid \Lambda^2) = \sum_{i,j=1}^{N} \underbrace{P(\Lambda_1 = i, \Lambda_2 = j)}_{=\lambda_i\lambda_j} \underbrace{H(Z'^{(1)} Z'^{(2)} \mid \Lambda_1 = i, \ldots}$$

channel, where 1st o/ from BSC(λ_i), 2nd c that of BSC(λ_j), their resp. input d (call them α_1, α_2).

Hence $H(Z^2) = nH(\Lambda) + H(Z'^2 \mid \Lambda^2)$

28/04/10 **LOOSENESS OF CBSC BOUNDS:**

$$\Pr\{ I(x;z) < n\,\mathrm{Cap}(\tau) - n\epsilon \}$$

$$\begin{aligned} I(x;z) &= H(z) - H(z|x) \\ &\geq n(1 + H(\lambda)) - \log\Big[N^n \sum_{w=0}^{n} B_w \rho^w\Big] \\ &\qquad - n\Big[H(\lambda) + \sum_{i=1}^{N} \lambda_i h(p_i \\ &= n\Big(1 - \sum_{i=1}^{N} \lambda_i h(p_i)\Big) - \log\Big[N^n \sum_{w=0}^{n} B_w \rho^w\Big] \end{aligned}$$

$$\Pr\{ I(x;z) < n\,\mathrm{Cap}(\tau) - n\epsilon \}$$

$$\leq \Pr\Big\{ \log\Big[N^n \sum_{w=0}^{n} B_w \rho^w\Big] > n\Big(1 - \sum_{i=1}^{N} \lambda_i h(p_i) - \mathrm{Cap}($$

$$\begin{aligned} \mathrm{Cap}(\tau) &= 1 + H(\underline{\lambda}) - \Big[H(\underline{\lambda}) + \sum_{i=1}^{N} \lambda_i h(p_i)\Big] \\ &= 1 - \sum_{i=1}^{N} \lambda_i h(p_i) \end{aligned}$$

Therefore

$$\Pr\{ I(x;z) < n\,\mathrm{Cap}(\tau) - n\epsilon \}$$

$$\leq \Pr\Big\{ \log\Big[N^n \sum_{w=0}^{n} B_w \rho^w\Big] > n\epsilon \Big\}$$

$$= \Pr\Big\{ N^n \sum_{w=0}^{n} B_w \rho^w > 2^{n\epsilon} \Big\}$$

Will need to back-off by ϵ bits, where $N = 2^\epsilon$
For $n = 4$, $\epsilon = 2$ bits (very loose bound)