CAPACITY OF THE $F_2 \times F_4$ CHANNEL

Let $P(x=0) = \alpha$, $P(x=1) = 1-\alpha$.

$$\Pr\{z=0\} = \alpha\,\lambda_1(1-p) + (1-\alpha)\lambda_1 p \qquad \alpha\lambda_1(1-2p)+\lambda_1 p$$
$$= \alpha[\lambda_1 - p\lambda_1 - p\lambda_1] + \lambda_1 p \triangleq c_0\alpha + d_0$$
$$\Pr\{z=1\} = \alpha\lambda_1 p + (1-\alpha)\lambda_1(1-p)$$
$$= \alpha\lambda_1[2p-1] + \lambda_1(1-p) \triangleq c_1\alpha + d_1$$
$$\Pr\{z=2\} = \alpha\lambda_2(1-q) + (1-\alpha)\lambda_2 q$$
$$= \alpha\lambda_2(1-2q) + \lambda_2 q \triangleq c_2\alpha + d_2$$
$$\Pr\{z=3\} = \alpha\lambda_2 q + (1-\alpha)\lambda_2(1-q)$$
$$= \alpha\lambda_2[2q-1] + \lambda_2[1-q] \triangleq c_3\alpha + d_3$$

$$H(Z) = -\sum_{i=0}^{3} (c_i\alpha + d_i)\log(c_i\alpha + d_i)$$

$$H(Z|X) = \alpha H(Z|X=0) + (1-\alpha) H(Z|X=1)$$
$$= H(Z|X=0) \quad \text{(independent of } \alpha\text{)}$$

$$\arg\max_{p(x)} I(X;Z) = \arg\max_{\alpha} H(Z)$$

$$\frac{dH(Z)}{dZ} = 0$$ (H(Z) is concave in p(x). Hence derivative = 0 will give a maxima (not a local max)

$$\Rightarrow -\sum_{i=0}^{3} c_i \log(c_i\alpha + d_i) - \sum_{i=0}^{3} c_i = 0$$
$$= 0$$

$$\Rightarrow \sum_{i=0}^{3} c_i \log(c_i\alpha + d_i) = 0.$$

... when $\alpha = 1/2$:

$$c_0 + 2d_0 = c_1 + 2d_1 = \lambda_1$$
$$c_2 + 2d_2 = c_3 + 2d_3 = \lambda_2$$

$$\sum c_i \log \frac{c_i + 2d_i}{2} \qquad c_0 \log\frac{\lambda_1}{2} + c_1 \log\frac{\lambda_1}{2}$$
$$+ (c_2 + c_3)\log\frac{\lambda_2}{2} = 0$$

> Unif distribution is capacity achieving

$$H(Z)\Big|_{\alpha=\frac{1}{2}} = -\frac{1}{2}\sum_{i=0}^{3}(c_i+2d_i)\log\left(\frac{c_i+2d_i}{2}\right)$$

$$= -\frac{1}{\cancel{2}}\left[\cancel{2}\lambda_1\log\frac{\lambda_1}{2} + \cancel{2}\lambda_2\log\frac{\lambda_2}{2}\right]$$

$$= -\lambda_1\log\lambda_1 - \lambda_2\log\lambda_2 + \lambda_1\log 2 + \lambda_2\log$$

$$= 1 + h(\lambda_1)$$

$$H(Z|X) = H(Z/X=0)$$

$$= -\Big[\lambda_1(1-p)\log\lambda_1(1-p) + \lambda_1 p\log\lambda_1 p + \lambda_2(1-q)\log\lambda_2(1-q) + \lambda_2 q\log\lambda_2 q\Big]$$

$$= -\lambda_1\big[(1-p)\log(1-p) + (1-p)\log\lambda_1 + p\log\lambda_1 + p\log$$

$$-\lambda_2\big[(1-q)\log(1-q) + (1-q)\log\lambda_2 + q\log\lambda_2 + q\log$$

$$= \lambda_1\big[h(p) - (1-\cancel{p})\log\lambda_1 - \cancel{p\log\lambda_1}\big]$$
$$+\lambda_2\big[h(q) - (1-\cancel{q})\log\underbrace{(1-\lambda_1)}_{\lambda_2} - \cancel{q\log\underbrace{(1-\lambda_1)}_{\lambda_2}}\big]$$

$$= \lambda_1 h(p) + \lambda_2 h(q) - \lambda_1(1-p)\log\lambda_1 - \lambda_1 p\log\lambda_1$$
$$-(1-\lambda_1)(1-q)\log(1-\lambda_1) - (1-\lambda_1)q\log(1-\lambda_1)$$

$$= \lambda_1 h(p) + \lambda_2 h(q) - \log\lambda_1\cdot\big[\lambda_1 - \cancel{\lambda_1}p + \cancel{\lambda_1}p\big]$$
$$-\log(1-\lambda_1)\big[(1-\lambda_1)(1-\cancel{q}+\cancel{q})\big]$$

$$= \lambda_1 h(p) + \lambda_2 h(q) + h(\lambda_1)$$

$$H(Z) - H(Z|X) = Cap(\tau)$$
$$= 1 - \lambda_1 h(p) - \lambda_2 h(q)$$

<u>Lemma 4</u> <u>in</u> <u>Pakzad-Shokrollahi</u>:

$$\Pr\{I(X;Z) < n\,Cap(\tau)\} \le n\,Cap(\tau) - \mathbb{E}[I(X;Z)]$$
$$= n[1+h(\lambda_1)] - \cancel{nH(Z|X)} - \mathbb{E}[H(Z)] + \mathbb{E}[\cancel{nH(Z/X)}]$$
$$= n[1+h(\lambda_1)] - \mathbb{E}[H(Z)]$$

$$H(Z) = -\sum_{i=0}^{3} (c_i \alpha + d_i) \log (c_i \alpha + d_i)$$

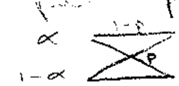
Define
$$c_0 = \lambda_1 c_0' \qquad d_0 = \lambda_1 d_0'$$
$$c_1 = \lambda_1 c_1' \qquad d_1 = \lambda_1 d_1'$$

$$c_2 = \lambda_2 c_2' \qquad d_2 = \lambda_2 d_2'$$
$$c_3 = \lambda_2 c_3' \qquad d_3 = \lambda_2 d_3'$$

$$H(Z) = -\sum_{i=0}^{1} \lambda_1 \left[(c_i' \alpha + d_i') \log (c_i' \alpha + d_i') + (c_i' \alpha + d_i') \log \lambda_1 \right.$$
$$- \sum_{i=2}^{3} \lambda_2 \left[(c_i' \alpha + d_i') \log (c_i' \alpha + d_i') + (c_i' \alpha + d_i') \log \lambda_2 \right.$$

<u>Check:</u>
$$\underbrace{c_0' \alpha + d_0'} + \underbrace{c_1' \alpha + d_1'} = \alpha [1-2p] + p + \alpha[2p-1] + (1-p) \qquad 1$$

These two quantities are the two output probs of a BSC with crossover prob. p



$$H(Z) = \lambda_1 \underset{\substack{\uparrow \\ BSC(p)}}{H(Z_1)} + \lambda_2 \underset{\substack{\uparrow \\ BSC(q)}}{H(Z_2)} + h(\lambda_1)$$

$$\Pr \{ I(X;Z) < n \, \mathrm{Cap}(\mathcal{C}) \}$$
$$\leq n - \mathbb{E}[H(Z)] + n h(\lambda_1)$$
$$= \lambda_1 [n - \mathbb{E}(H(Z_1))] + \lambda_2 [n - \mathbb{E}(H(Z_2))] + n h(\lambda_1) - n h(\lambda_1)$$
$$\lambda_1 [n - \mathbb{E}(H(Z_1))] + \lambda_2 [n - \mathbb{E}(H(Z_2))] \quad (1)$$

$$\leq \lambda_1 \log_2 \left(\sum_{w=0} \mathbb{E}[B_w] (1-2p)^{2w} \right) + \lambda_2 \log_2 \left(\sum_{w=0}^{n} \mathbb{E}[B_w] (1-2q)^{2w} \right)$$

$\mathbb{E}[\log X] \le \log \mathbb{E}(x)$ $\qquad (1-x)^k$

$$\le \log_2 \left[\sum_{w=0}^{n} \mathbb{E}(B_w) \left(\lambda_1 (1-2p)^{2w} + \lambda_2 (1-2q)^{2w} \right) \right]$$

$$\lambda_1 (1-2p)^{2w} + \lambda_2 (1-2q)^{2w} \overset{?}{=} (1-2x)^{2w}$$

(Don't think this can be solved for all w).

① can be interpreted as giving rise to two th

PINSKERS INEQUALITY FOR THE BSC:

$$\underline{x} \in \mathbb{F}_2^k \to \boxed{\mathcal{C}} \xrightarrow{\underline{y} \in \mathcal{C}} \boxed{BSC} \to \underline{z} \in \mathbb{F}_2^n$$

$$I(\underline{x};\underline{z}) \ge \frac{\log e}{2} \left(\sum_{\substack{\underline{x} \in \mathbb{F}_2^k \\ \underline{z} \in \mathbb{F}_2^n}} \left| f(\underline{x},\underline{z}) - f(\underline{x}) f(\underline{z}) \right| \right)^2$$

$$f(\underline{x},\underline{z}) = f(\underline{x})\, f(\underline{z} \mid \underline{x})$$
$$= f(\underline{x})\, f(\underline{z} \mid \underline{y}) = \frac{1}{2^k} \cdot p^{d_H(\underline{y},\underline{z})} (1-p)^{n-d_H}$$

$$f(\underline{z}) = \frac{1}{2^k} \sum_{\underline{y} \in \mathcal{C}} p^{d_H(\underline{y},\underline{z})} (1-p)^{n-d_H(\underline{y},\underline{z})}$$

$$I(\underline{x};\underline{z}) \ge \frac{\log e}{2^{k+1}} \left(\sum_{\substack{\underline{y} \in \mathcal{C} \\ \underline{z} \in \mathbb{F}_2^n}} \left| p^{d_H(\underline{y},\underline{z})} (1-p)^{n-d_H(\underline{y},\underline{z})} - \frac{1}{2^k} \sum_{\underline{y}' \in \mathcal{C}} p^{d_H(\underline{y}',\underline{z})} (1- \right. \right.$$

USING WADAYAMA'S COSET WEIGHT DISTRIB.

MacMullan & Collins:

# of n-tuples j in row i of $\triangleq A_j(\underline{s})$

$$H(R) = -\sum_{i=1}^{2^{n-k}} \sum_{j=0}^{n} w(i,j)\, p^j (1-p)^{n-j} \log \underbrace{\left[\sum_{j=0}^{n} w(i,j)\, p^j (1-p)^{n-j} \right.}_{\triangleq A}$$

Constant row weight ensemble (Wadayama, Lemma 2):

takes on all 2^{n-k} values of a binary vector of length $n-k$

$$\tilde{A}_w(\underline{s}) = [\nu(n,d,w)]^{n-k-|\underline{s}|} \, [1-\nu(n,d,w)]^{|\underline{s}|} \binom{n}{w}$$ $\quad \Big\{ m \equiv n-$ rows of H

$\underline{s} \to$ syndrome $\quad |\underline{s}| = w_H(\underline{s}) \quad H \to m \times n \quad d \to$ constant

$$\gamma(n,d,w) = \frac{1}{\binom{n}{d}} \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{w}{2i} \binom{n-w}{d-2i}$$

Let $\underline{s}_i \rightarrow$ syndrome corresponding to i^{th} row

$$A = \sum_{j=0}^{n}$$

CONVEX COMBINATION OF SEVERAL BSCs :

$$\phi = \sum_{i=1}^{N} \left(p_i \, \Delta_{\log \frac{p_i}{1-p_i}} + (1-p_i) \, \Delta_{-\log \frac{p_i}{1-p_i}} \right) \lambda_i \quad , \quad \sum_{i=1}^{N} \lambda_i$$

This gives a physical channel that is a straightf extension of the 2-BSC case (can verify that the LLRs are identical)

Call the outputs (Z) of the 2N-ary channel $\{0_i, 1_i\}$

$$P(Z = 0_i) = \alpha \lambda_i (1-p_i) + (1-\alpha) \lambda_i p_i$$
$$= \alpha \lambda_i (1-2p_i) + \lambda_i p_i \triangleq c_i \alpha + d_i$$
$$P(Z = 1_i) = \alpha \lambda_i p_i + (1-\alpha) \lambda_i (1-p_i)$$
$$= \alpha \lambda_i (2p_i - 1) + \lambda_i (1-p_i) \triangleq e_i \alpha + f_i$$

$$H(Z) = - \sum_{i=1}^{N} \Big[(c_i \alpha + d_i) \log (c_i \alpha + d_i) + (e_i \alpha + f_i) \log (e_i \alpha + f_i)$$

Let $c_i = \lambda_i c_i'$, $d_i = \lambda_i d_i'$, $e_i = \lambda_i e_i'$, $f_i = \lambda_i f_i'$

$$H(Z) = - \sum_{i=1}^{N} \lambda_i \Big[(c_i'\alpha + d_i') \log (c_i'\alpha + d_i') + (e_i'\alpha + f_i') \log (e_i'\alpha + f_i')$$
$$+ (\alpha c_i' + d_i') \log \lambda_i + (e_i'\alpha + f_i') \log \lambda_i \Big]$$

$$c_i'\alpha + d_i' = \alpha (1-2p_i) + p_i$$
$$e_i'\alpha + f_i' = \underbrace{\alpha (2p_i - 1) + (1-p_i)}_{\text{(sums to 1)}}$$

o/p probs of a BSC (p_i)

$$H(Z) = \left[\sum_{i=1}^{N} \lambda_i H(Z_i) \right] + H(\{\lambda_1, \lambda_2, \ldots, \lambda_n\})$$

INTERFERENCE ALIGNMENT : Tx BEAMFORMER SEL

$$\sum_{j\neq k} \| H_{ki} \hat{V}_k , U_i \|_M^2 + w \| H_{kk} \hat{V}_k , U_k^{\perp} \|_M^2$$

$$= I - U_k U_k^{\dagger}$$

$$= \sum_{j\neq k} \| H_{ki} \hat{V}_k - U_i U_i^{\dagger} H_{ki} \hat{V}_k \|_F^2 + w \| H_{kk} \hat{V}_k - \underbrace{U_k^{\perp} (U_k^{\perp})^{\dagger}}_{= I - U_k U_k^{\dagger}} H$$

$$= \sum_{j\neq k} \mathrm{tr} \left\{ (\hat{V}_k^{\dagger} H_{ki}^{\dagger} - \hat{V}_k^{\dagger} H_{ki}^{\dagger} U_i U_i^{\dagger}) (H_{ki} \hat{V}_k - U_i U_i^{\dagger} H_{ki} \hat{V}_k) \right\}$$

$$+ w \| U_k U_k^{\dagger} H_{kk} \hat{V}_k \|_F^2$$

$$= \sum_{j\neq k} \mathrm{tr} \left\{ \hat{V}_k^{\dagger} H_{ki}^{\dagger} H_{ki} \hat{V}_k - V_k^{\dagger} H_{ki}^{\dagger} U_i U_i^{\dagger} H_{ki} \hat{V}_k - \cancel{\hat{V}_k^{\dagger} H_{ki}^{\dagger} U_i U_i^{\dagger} H} \right.$$

$$\left. + \cancel{\hat{V}_k^{\dagger} H_{ki}^{\dagger} U_i U_i^{\dagger} U_i U_i^{\dagger} H_{ki} \hat{V}_k} \right\}$$

$$+ w\, \mathrm{tr} \left\{ \hat{V}_k^{\dagger} H_{kk}^{\dagger} U_k \cancel{U_k^{\dagger}} \cancel{U_k} U_k^{\dagger} H_{kk} \hat{V}_k \right\}$$

$$= \mathrm{tr} \left\{ \hat{V}_k^{\dagger} \left[\sum_{j\neq k} H_{ki}^{\dagger} (I - U_i U_i^{\dagger}) H_{ki} + w H_{kk}^{\dagger} U_k U_k^{\dagger} H_{kk} \right. \right.$$

18|12|09

PROOF OF THEOREM 6, [PAKZAD-SHOKROLLAHI]

$a = T/n \to \alpha$

$A : T \times n$

$x = \dfrac{nm}{n}$

$\phi = n/k$

$G^T : n \times k$

$x = dw/k$ ~~(have changed this from the version the paper)~~

$$\binom{T}{m} = \frac{an}{\sqrt{2\pi x (an - x) an}} \left(\frac{(an)^{an} (an - x)^x}{x^x (an - x)^{an}} \right)^{n/n} (1 + o(1))$$

$$\downarrow$$

$q = \dfrac{a-x}{a}$

$$\binom{n}{w} = \frac{\phi d}{\sqrt{2\pi x (\phi d - x) \phi k}} \left(\frac{(\phi d)^{\phi d} (\phi d - x)^x}{x^x (\phi d - x)^{\phi d}} \right)^{k/d} (1 + o(1))$$

$p = \dfrac{nm}{T} = \dfrac{nx}{T} = x/a$

$an = T/n \cdot n$

$= T/x \cdot \dfrac{xx}{m}$

$$x' = x/d \qquad \binom{n}{w} = \frac{\phi d}{\sqrt{2\pi d x' (\phi d - d x') \phi k}} \left(\frac{(\phi d)^{\phi d} (\phi d - d x')^{d x'}}{(x' d)^{x' d}} \right.$$

Set $p = \frac{nm}{T}$ (Kolchin's notation)

$$= \frac{nx}{T} = \frac{x}{a}$$

$$q = 1 - \frac{x}{a} = \frac{a-x}{a}$$

$$\binom{T}{m} = \frac{1}{p^m q^{T-m}\sqrt{2\pi T p q}}\,(1+o(1))$$

$$= \frac{\cancel{a^m}\; a^{T-m}}{x^m (a-x)^{T-m}\sqrt{2\pi \cancel{a} n \frac{x(a-x)}{a^2}}}\,(1+o(1))$$

$$= \frac{a^{an}\sqrt{a}\,(a-x)^m}{x^m (a-x)^{an}\sqrt{2\pi x(a-x)n}}\,(1+o(1))$$

Set $p = \frac{m}{Tn} = \frac{nx}{Tn^2} = \frac{x}{an^2}$

$$\binom{n}{w} = \frac{\phi}{\sqrt{2\pi \frac{x}{d}(\phi - \frac{x}{d})\phi k}}\left(\frac{\phi^\phi \cancel{d^\phi}\, d^{x/d}(\phi - x/d)^{x/d}}{x^{x/d}\,\cancel{d^\phi}(\phi - x/d)^\phi}\right)^k (1+o(1)$$

$$= \frac{\phi}{\sqrt{2\pi x'(\phi - x')\phi k}}\left(\frac{\phi^\phi (\phi - x')^{x'}}{x'^{x'}(\phi - x')^\phi}\right)^k (1+o(1))$$

where $x' = x/d = \frac{\lambda \tanh\lambda}{d}$ $= w/k$

$\Rightarrow wd = k\lambda\tanh\lambda$

$\Rightarrow w = \frac{k\lambda}{d}\tanh\lambda$

$$E[B_w](1-2p)^{2w} \le 2\binom{n}{w}\cosh(\lambda)^k\left(\frac{wd}{\lambda k e}\right)^{wd} wd\,(1-p)^{2w}$$

$$\frac{2\phi}{\sqrt{2\pi x'(\phi - x')\phi k}}\left(\frac{\phi^\phi(\phi - x')^{x'}}{x'^{x'}(\phi - x')^\phi}\right)^k \cosh(\lambda)^k\left(\frac{wd}{\lambda k e}\right)^{wd} wd\,($$

$(1+$

$$= \frac{wd\phi}{\sqrt{2\pi x'(\phi - x')\phi k}} \cdot \left[\cosh\lambda \; \phi^{\phi} \frac{(d\phi - \lambda\tanh\lambda)^{\frac{\lambda\tanh\lambda}{d}}}{\cancel{d^{(\lambda\tanh\lambda)/d}}} \; \frac{\cancel{d}^{(\lambda t}}{(\lambda\tanh\lambda)^{\lambda\tan}} \right.$$

$$\left. \cdot \frac{d^{\phi}}{(d\phi - \lambda\tanh\lambda)^{\phi}} \right]^{k} \cdot \left[\frac{2(wd)^{wd}}{(\lambda k e)^{wd}} \; (1-2p)^{2w} \right] \; (1+$$

$$= \frac{wd\phi}{\sqrt{2\pi x'(\phi - x')\phi k}} \left[\cosh\lambda \left(\frac{d\phi}{d\phi - \lambda\tanh\lambda} \right)^{\phi} \left(\frac{d\phi - \lambda\tanh\lambda}{\lambda\tanh\lambda} \right)^{\frac{\lambda\tan}{d}} \right.$$

$$\left[\frac{(\cancel{k}\cancel{\lambda}\tanh\lambda)^{\lambda\tanh\lambda}}{(\cancel{\lambda}\cancel{k}e)^{\lambda\tanh\lambda}} \cdot (1-2p)^{2\lambda(\tanh\lambda)/d} \cdot 2^{1/k} \right]^{k}$$

$$= \frac{2wd\phi}{\sqrt{2\pi x'(\phi - x')\phi k}} \left[\cosh\lambda \left(\frac{\tanh\lambda}{e} \right)^{\lambda\tanh\lambda} \left(\frac{d\phi}{d\phi - \lambda\tanh\lambda} \right)^{\phi} \right.$$

$$\left(\frac{d\phi - \lambda\tanh\lambda}{\lambda\tanh\lambda} (1-2p)^{2} \right)^{\frac{\lambda\tanh\lambda}{d}}$$

$$\cdot \left(1 + o(1)\right)$$

Proof of (5): [Kolchin, pg. 162]

(Assume $T > n$ (i.e., $\alpha > 1$))

$$\sum_{m=T-m_0}^{T} \binom{T}{m} P_E(nm, n) \; (1-2p)^{2m}$$

$$\leq c(nT)^{1/2} q^{n} \sum_{m=T-m_0}^{T} \binom{T}{m} \frac{q^{m}(1-q)^{T-m}}{q^{T}(1-q)^{m_0}} (1-2p)^{2}$$

$$\leq c(nT)^{1/2} (1-2p)^{2(T-m_0)} \cdot \frac{q^{n-T}}{(1-q)^{m_0}}$$

$$= \frac{c(nT)^{1/2}}{(1-q)^{m_0}} \; \frac{(1-2p)^{2(T-m_0)}}{q^{T-T/\alpha}} = \frac{c(nT)^{1/2}}{(1-q)^{m_0}} \; \frac{(1-2p)^{2(T}}{\left[q^{(1-1/\alpha}\right.}$$

$\alpha = T/N > 1 \Rightarrow 1/\alpha < 1$. $m_0 \leq \delta T$, δ may be chosen a

$$= \frac{c\,(\pi T)^{1/2}}{(1-q)^{m_0}} \cdot \left[\frac{(1-2p)^{2\left(1-\overbrace{\left(\frac{m_0}{T}\right)}^{\delta}\right)}}{q^{\left(1-\frac{1}{\alpha}\right)}} \right]^T$$

Can consider δ such that δT is an integer, whi[ch] we may equate to m_0. Above converges to zer[o if]

$$(1-2p)^{2(1-\delta)} \overset{?}{<} q^{1-1/\alpha}$$

$$= \left(\frac{1+e^{-2\lambda}}{2} \right)^{\left(1-\frac{1}{\alpha}\right)}$$

$$\lambda = \frac{\pi m}{n} \quad : \quad \frac{(T-m_0)\,\pi}{n} \le \lambda \le \frac{T\pi}{n}$$

$$\left(\frac{T}{n} - \frac{m_0}{n}\right)\pi \le \lambda \le \frac{T}{n}\pi$$

As $T, n \to \infty$, $\lambda \to \alpha\pi$

$$(1-2p)^{2(1-\delta)} \overset{?}{<} \left(\frac{1+e^{-2\alpha\pi}}{2} \right)^{\left(1-\frac{1}{\alpha}\right)}$$

$$\iff \quad p > \frac{1}{2}\left\{ 1 - \left(\frac{1+e^{-2\alpha\pi}}{2} \right)^{\frac{(1-1/\alpha)}{2(1-\delta)}} \right\} \triangleq f(\alpha)$$

If $\alpha \to 1$, then RHS becomes $0 \Rightarrow$ convergence for all $p > 0$. For every p, can we identify an α such that the above is satisfied?
As $\alpha \uparrow$, the above RHS $\uparrow$. If p is fixed can $\uparrow \alpha$ till the above RHS hits p. Hence p determines an α-region of convergence

$$h(\alpha) = \frac{1}{2}\left\{ 1 - \left(\frac{1+e^{-2\alpha\pi}}{2} \right)^{\frac{1}{2}(1-1/\alpha)} \right\} \quad \left(\text{taking lim } \delta \to 0\right)$$

Determine the max alpha s.t. $h(\alpha) < p$

$$h(\alpha) \longrightarrow \frac{1}{2}\left(1-\sqrt{.5}\right) = 0.1464 \quad \text{as} \quad \alpha \to \infty$$

Hence if $p \geq 0.1464$, the max α from the above bound is ∞. Will obtain a non-infinite soln. the max α if $p < 0.1464$

KOLCHIN'S ENSEMBLE FOR AWGN : (contd. from previous

Last part of the sum:

$$\sum_{w=n-w_0}^{n} B_w e^{-\rho w} \leq c\,(dn)^{1/2}\, q^{n-k}\, e^{-\rho(n-w_0)} \sum_{w=n-w_0}^{n} \binom{n}{w} \frac{q^w (1-q)^{n-w}}{q^n (1-q)^{w_0}}$$

$$\leq c\,(dn)^{1/2}\, e^{-\rho(n-w_0)}\, q^{\cancel{n}-k} \cdot \frac{q^{-\cancel{n}}}{(1-q)^{w_0}}$$

$$= \frac{c\,(dn)^{1/2}}{(1-q)^{w_0}}\, q^{-Rn}\, e^{\rho w_0} \cdot e^{-\rho n}$$

$(q^{-R} = (e^{\ell}$

$$= \frac{c\,(dn)^{1/2}\, e^{\rho w_0}}{(1-q)^{w_0}} \left[e^{-\rho}\, e^{-R \ln q} \right]^n$$

$$= \frac{c\,(dn)^{1/2}\, e^{\rho w_0}}{(1-q)^{w_0}} \left[e^{-\rho - R \ln q} \right]^n$$

Need $\frac{1}{2^k} \sum_{w=n-w_0}^{n} B_w e^{-\rho w}$ to vanish. $2^{-k} = 2^{-Rn} = (e^{-R \ln 2})$

$$-\rho - R \ln(2q) < 0$$

$$q = \left(\frac{1+e^{-2\lambda}}{2} \right) \quad ; \quad \lambda = \frac{dw}{n-k}$$

$$\frac{d(n-w_0)}{n-k} \leq \lambda \leq \frac{dn}{n-k}$$

$$\lambda \to \frac{dn}{n-k} \quad \text{as} \quad n, k \to \infty$$

$$\rho > -R \ln \left(\cancel{2} \cdot \frac{1+e^{-2(dn/n-k)}}{\cancel{2}} \right)$$

$$\rho > -R \ln\left[1 + e^{\frac{-2d}{1-R}} \right] \quad \text{—— (1)}$$

For $R > 0$, $1 + e^{-2d/1-R} < 1 \Rightarrow$ RHS above is

In the region defined by (1), the last part of the sum vanishes.

Middle part of the sum

{ From Pakzad's paper eq (7) by replacing k with n-k

$$\frac{1}{2^k} B_w e^{-\rho w} \leq \left(2^{-R} \left(\frac{ne}{w}\right)^{w/n} \cosh(\lambda)^{(n-k)/n} \left(\frac{wd}{\lambda(n-k)e}\right)^{wd/n} e^{-\rho w/n} \underbrace{\sqrt{wd}}_{\to 1 \text{ as } n \to \infty} \right)^n$$

$x = w/k$, $\phi = n/k = \frac{1}{R}$

Need to show

$$-\frac{1}{\phi}\ln 2 + \frac{x}{\phi} \ln\left(\frac{\phi e}{x}\right) + \left(\frac{\phi - 1}{\phi}\right) \ln \cosh \lambda + \frac{xd}{\phi} \ln\left(\frac{d}{\lambda e\left(\frac{\phi}{x} - \frac{1}{x}\right)}\right) - \frac{\rho x}{\phi} < 0$$

$$x = \frac{\lambda \tanh \lambda}{d}:$$

$$-\ln 2 + \frac{\lambda \tanh \lambda}{d} \ln\left(\frac{\phi e d}{\lambda \tanh \lambda}\right) + (\phi - 1) \ln \cosh \lambda + \lambda \tanh \lambda \ln\left(\frac{\lambda \tanh \lambda}{\lambda e(\phi - 1)}\right) - \rho \cdot \frac{\lambda \tanh \lambda}{d} <$$

$$\rho > \frac{d}{\lambda \tanh \lambda}\left[-\ln 2 + \frac{\lambda \tanh \lambda}{d} \ln\left(\frac{\phi e d}{\lambda \tanh \lambda}\right) + (\phi - 1) \ln \cosh \lambda + \lambda \tanh \lambda \ln\left(\frac{\tanh \lambda}{e(\phi - 1)}\right) \right] = f(\lambda)$$

If $\phi = \frac{1}{R}$ is fixed, this gives us a P-ROC.

$$\rho > \max_{\lambda > 0} f(\lambda)$$

COMPUTING Ω_2 THRESHOLDS :

[Etesami - Shokrollahi] :

$$\Omega_2 \geq \frac{\pi(\mathcal{C})}{2} = \frac{\mathrm{Cap}(\mathcal{C})}{2\,\mathbb{E}[\mathcal{C}]}$$

Know: $\pi(BSC(\epsilon)) = \dfrac{1-h(\epsilon)}{(1-2\epsilon)^2}$

$$\Rightarrow \mathbb{E}[BSC(\epsilon)] = (1-2\epsilon)^2 \qquad \left\{ \equiv \mathbb{E}\left[\tanh\left(\tfrac{Z}{2}\right)\right] \right.$$

where Z is the o/p BSC(ϵ)

Consider the convex combination of BSCs:

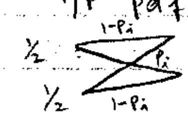
$$\mathbb{E}[CBSC(\underline{p}, \underline{\lambda})] = \int_{-\infty}^{\infty} \tanh\left(\frac{x}{2}\right) \underbrace{g(x)}_{\text{pdf of the LLRs for channel}}\, dx$$

$$= \int_{-\infty}^{\infty} \tanh\left(\frac{x}{2}\right) \sum_{i=1}^{N} \lambda_i \underbrace{\phi_{p_i}(x)}_{\text{pdf of the LLR for } BSC(p_i)}\, dx$$

$$= \sum_{i=1}^{N} \lambda_i\, \mathbb{E}[BSC(p_i)] = \sum_{i=1}^{N} \lambda_i (1-2p_i)^2$$

$$\mathcal{C}[CBSC(\underline{p}, \underline{\lambda})] = \sum_{i=1}^{N} \lambda_i \underbrace{H(Z_i)}\Big|_{\alpha=1/2} - \sum_{i=1}^{N} \lambda_i h(p_i)$$

$Z_i \to$ o/p of a BSC(p_i) with i/p pdf $\{\alpha, 1-\alpha\}$



$$= \sum_{i=1}^{N} \lambda_i [1 - h(p_i)] = \sum_{i=1}^{N} \lambda_i \mathcal{C}[\ldots$$

Hence for $CBSC(\underline{\rho}, \underline{\lambda})$,

$$\Pi[CBSC(\underline{\rho},\underline{\lambda})] = \frac{\sum_{i=1}^{N} \lambda_i [1 - h(p_i)]}{\sum_{i=1}^{N} \lambda_i (1-2p_i)^2}$$

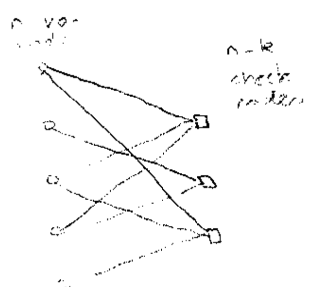
and $\Omega_2 \geq \frac{\Pi[CBSC(\underline{\rho},\underline{\lambda})]}{2}$

Computing Ω_2 for our LDGM construction.

$G^T_{n\times k}$ corresponds to Kalai's $A_{T\times n}$

$\Rightarrow$ upto r ones in every row of G^T
column of G

Bipartite graph for codes



H

$$\equiv \begin{bmatrix} & & \end{bmatrix}_{(n-k)\times n}$$

No of ones in each row of H corresponds to the output degree

9|01|10

GAUSSIAN CASE

Claim $\quad |\underline{z} - s_i|^2 + |\underline{z} - s_j|^2 = 2\left|\underline{z} - \frac{s_i + s_j}{2}\right|^2 + \frac{|s_i - s_j|^2}{2}$

LHS $\quad |\underline{z}|^2 + |s_i|^2 - \underline{z}^\dagger s_i - s_i^\dagger \underline{z} + |\underline{z}|^2 + |s_j|^2 - \underline{z}^\dagger s_j - \dots$

$\quad = 2|\underline{z}|^2 + |s_i|^2 + |s_j|^2 - (\underline{z}^\dagger + \underline{z}^\dagger) s_i - (\underline{z}^\dagger + \underline{z}^\dagger) s_j$

$\quad = 2|\underline{z}|^2 + |s_i|^2 + |s_j|^2 - (\underline{z}^\dagger + \underline{z}^\dagger)(s_i + s_j)$

RHS $\quad 2|\underline{z}|^2 + 2\frac{|s_i + s_j|^2}{4} - 2 \cdot \underline{z}^\dagger\left(\frac{s_i + s_j}{2}\right) - 2\left(\frac{s_i + s_j}{2}\right)^\dagger \underline{z}$

$\quad + \frac{|s_i - s_j|^2}{2}$

$$-\log\left[\frac{\cancel{\rho^{n/2}} \cdot 2 \cancel{\pi}}{|c|\, 2^n \cancel{\pi^{n/2}}\, \cancel{\rho^{n/2}}}\right] \qquad e^{\frac{2}{}}\ \cancel{2}|c_i|^2 + \cancel{2}|c_j|^2 - \cancel{2}|c_i + c_j|^2 \qquad \frac{|c_i|^2 + |c_j|^2 - c_i c_j - c_j c_i}{2} = \frac{|c_i - c_j|^2}{2}$$

$$e^{-\frac{1}{2}\left[z - \frac{c_i + c_j}{2}\right] \underbrace{\cancel{4}\, 2\rho I}_{K^{-1} = 2\rho I} \left[z - \frac{c_i + c_j}{2}\right]}$$

$$K^{-1} = 2\rho I \Rightarrow K = \frac{1}{2\rho} I \qquad \det(K) = \frac{1}{(2\rho)^n}$$

$$= 2|z|^2 + \frac{\cancel{2}|c_i|^2 + \cancel{2}|c_j|^2}{\cancel{2}} - (z^{+} + z^{T})(c_i + c_j)$$

$= LHS.$

FIXING A BUG IN THE GAUSSIAN CASE

From the previous claim, we may rewrite (3) of the Gaussian writeup as

$$\int p^2(z)\, dz = \frac{\rho^n}{|C|^2 (2\pi)^n} \sum_{i,j=1}^{|C|} \underbrace{\int \exp\left\{-\frac{\rho}{2}\left[2\left|z - \frac{c_i + c_j}{2}\right|^2 + \frac{|c_i - c_j|^2}{2}\right]\right\}}_{\triangleq B}$$

$$B = e^{-\rho/4 |c_i - c_j|^2} \cdot \frac{\int e^{-\rho\left|z - \frac{c_i + c_j}{2}\right|^2} dz}{\sqrt{(2\pi)^n |K|}} \cdot \sqrt{(2\pi)^n |K|}$$

$$\left\{ K = \frac{1}{2\rho} I \right\}$$

$$= \sqrt{\frac{(\cancel{2}\pi)^n}{(\cancel{2}\rho)^n}}\ e^{-\rho/4 |c_i - c_j|^2}$$

$$\therefore \int p^2(z)\, dz = \frac{\rho^n}{|C|^2 (2\pi)^n} \sqrt{\frac{\pi^n}{\rho^n}} \sum_{i,j=1}^{|C|} e^{-\rho/4 |c_i - c_j|^2}$$

$$= \frac{\rho^{n/2}}{|C|^2\, 2^n\, \pi^{n/2}} \sum_{i,j=1}^{|C|} e^{-\rho/4 |c_i - c_j|^2}$$

$$= \underbrace{\frac{\rho^{n/2}}{|C|^2\, 2^n\, \pi^{n/2}} \sum_{w=0}^{n} B_w e^{-\rho w}}_{P}$$

{ following same steps as in document