$$\frac{\cancel{2}|\varepsilon_i|^2 + \cancel{2}|\varepsilon_j|^2 - \cancel{2}|\varepsilon_i+\varepsilon_j|^2}{\ldots} = \frac{|\varepsilon_i|^2+|\varepsilon_j|^2 - \varepsilon_i\varepsilon_j - \varepsilon_j\varepsilon_i}{2} = \frac{|\varepsilon_i-\varepsilon_j|^2}{2}$$

$$e^{-\frac{1}{2}\left[z-\frac{\varepsilon_i+\varepsilon_j}{2}\right]\underbrace{2\rho I}\left[z-\frac{\varepsilon_i+\varepsilon_j}{2}\right]}$$

$K^{-1} = 2\rho I \Rightarrow K = \frac{1}{2\rho} I \quad \det(K) = \frac{1}{(2\rho)^n}$

$$= 2|\underline{z}|^2 + \frac{\cancel{2}|\varepsilon_i|^2+\cancel{2}|\varepsilon_j|^2}{\cancel{2}} - (\underline{z}^+ + \underline{z}^T)(\varepsilon_i+\varepsilon_j)$$

$= LHS.$

FIXING A BUG IN THE GAUSSIAN CASE

From the previous claim, we may rewrite (3) of the Gaussian writeup as

$$\int p^2(\underline{z})\,d\underline{z} = \frac{\rho^n}{|C|^2(2\pi)^n}\sum_{i,j=1}^{|C|}\underbrace{\int \exp\left\{-\frac{\rho}{2}\left[2\left|\underline{z}-\frac{\varepsilon_i+\varepsilon_j}{2}\right|^2 + \frac{|\varepsilon_i-\varepsilon_j|^2}{2}\right]\right\}}_{\triangleq B}$$

$$B = e^{-\rho/4|\varepsilon_i-\varepsilon_j|^2} \cdot \frac{\int e^{-\rho\left|\underline{z}-\frac{\varepsilon_i+\varepsilon_j}{2}\right|^2} d\underline{z}}{\sqrt{(2\pi)^n|K|}} \cdot \sqrt{(2\pi)^n|K|}$$

$\{K = \frac{1}{2\rho} I\}$

$$= \sqrt{\frac{(\cancel{2}\pi)^n}{(\cancel{2}\rho)^n}}\; e^{-\rho/4|\varepsilon_i-\varepsilon_j|^2}$$

$$\therefore \int p^2(\underline{z})\,d\underline{z} = \frac{\rho^n}{|C|^2(2\pi)^n}\sqrt{\frac{\pi^n}{\rho^n}}\sum_{i,j=1}^{|C|} e^{-\rho/4|\varepsilon_i-\varepsilon_j|^2}$$

$$= \frac{\rho^{n/2}}{|C|^2 2^n \pi^{n/2}}\sum_{i,j=1}^{|C|} e^{-\rho/4|\varepsilon_i-\varepsilon_j|^2}$$

$$= \underbrace{\frac{\rho^{n/2}}{|C|2^n\pi^{n/2}}\sum_{w=0}^{n} B_w e^{-\rho w}}_{P}$$

{following same steps as in document

$$I(\underline{x};\underline{z}) \geq -\log P - \frac{1}{2}\log\frac{(2\pi e)^n}{\cdots}$$

$$= -\log \frac{e^{n/2}}{|C|\,2^{n/2}} \sum_{w=0}^{n} B_w e^{-\rho w}$$

$$\boxed{I(\underline{x};\underline{z}) \geq -\log \frac{(e/2)^{n/2}}{|C|} \sum_{w=0}^{n} B_w e^{-\rho w}}$$

$$\Pr\{ I(\underline{x},\underline{z}) < n\,\mathrm{Cap}(C) - \epsilon n \}$$

$$\leq \Pr\left\{ -\log \frac{(e/2)^{n/2}}{|C|} \sum_{w=0}^{n} B_w e^{-\rho w} < n\,\mathrm{Cap} - n\epsilon \right\}$$

$$= \Pr\left\{ \frac{2^{Rn}}{(e/2)^{n/2}} \cdot \frac{1}{\sum_{w=0}^{n} B_w e^{-\rho w}} < 2^{n\,\mathrm{Cap}(C) - n\epsilon} \right\}$$

$$= \Pr\left\{ \sum_{w=0}^{n} B_w e^{-\rho w} > \frac{2^{Rn}}{(e/2)^{n/2}} \cdot \frac{1}{2^{n\,\mathrm{Cap}(C) - n\epsilon}} \right\}$$

$$= \Pr\left\{ \left(\sqrt{e/2} \cdot \frac{1}{2^R}\right)^n \sum_{w=0}^{n} B_w e^{-\rho w} > 2^{-n\,\mathrm{Cap}(C) + n\epsilon} \right\}$$

$$= \Pr\left\{ \underbrace{\left(\sqrt{\frac{e}{2}} \cdot \frac{2^{\mathrm{Cap}(C)}}{2^R}\right)^n \sum_{w=0}^{n} B_w e^{-\rho w}}_{\triangleq D} > 2^{n\epsilon} \right\}$$

Let R denote the ρ-region where $D \to 0$. The code achieves capacity in this region! Need to investigate the behaviour of D

Have investigated the behaviour of $\frac{1}{2^n}\sum B_w e^{-\rho w}$ pages before. Will now modify that to study

NOTE: Due to looseness of our bounds, need to backoff by $\epsilon > 0$ capacity. Set $R = Cap(C)$, and $\sqrt{\frac{e}{2}} = 2^{\epsilon}$

$$\Rightarrow \epsilon = \log_2\left(\sqrt{\frac{e}{2}}\right) = 0.2213 \text{ bits}$$

<u>1st part of the sum</u> (<u>from previous NB</u>):

<u>NOTE</u>: For $R < Cap(C)$, and since $2 < e$ we have that $\sqrt{\frac{e}{2}} \, \frac{2^{Cap(C)}}{2^R} > 1$

Let $b = \frac{2^{Cap(C)}}{2^R}$, after backing off by ϵ bits

$$b^n \sum_{w=0}^{n} B_w e^{-Pw} \leftarrow \text{Need to analyse.}$$

$$b^n \sum_{1 \le w \le \delta n} \binom{n}{w} P_E(dw, n-k) \, e^{-wP}$$

$$\le b^n \sum_{1 \le w \le \delta n} \left[\left(\frac{1}{1-R}\right)^{d/2} d^{d/2 - 1} e^{4d} \delta^{d/2-1} e^{-P} \right]$$

$\boxed{\text{CHECK}}$

$$\le \epsilon \text{ for some } \delta > 0$$

<u>Last part of the sum</u> (<u>from 3 pages before</u>)

$$b^n \sum_{w=n-w_0}^{n} B_w e^{-Pw} \le \frac{c\,(dn)^{1/2} e^{Pw_0}}{(1-q)^{w_0}} \left[b \, e^{-P - R\ln q} \right]^n$$

$$= \frac{c\,(dn)^{1/2} e^{Pw_0}}{(1-q)^{w_0}} \left[e^{\ln b - P - R\ln q} \right]^n$$

For the above to vanish,

$$\ln b - P - R\ln q < 0 \quad \text{— ①}$$

$H \cdot (n-k) \times$
$H^T : n \times$

$$q \ge \left(\frac{1 + e^{-2\lambda}}{2}\right) \quad ; \quad \lambda = \frac{dw}{n-k}$$

{ can replace q with $\frac{1+e^{-2\lambda}}{2}$ in ①, this does not affect? the result }

$$\frac{d(n-w_0)}{n-k} \le \lambda \le \frac{dn}{n-k}$$

$$\Rightarrow \lambda \to \underbrace{\frac{dn}{n-k}}_{d/(1-R)} \text{ as } n, k \to \infty$$

Have corrected this, FOLLOW UP!

$$\rho > \ln b - R \ln\left(\frac{1+e^{-2d/(1-R)}}{2}\right)$$

$$\boxed{\rho > \ln 2\left[\mathrm{Cap}(C) - R\right] - R\ln\left(\frac{1+e^{[illegible]}}{2}\right)}$$

Middle part of the sum (from 3 pages back)

$$b^n B_w e^{-\rho w} \le \left(b \left(\frac{ne}{w}\right)^{w/n} (\cosh\lambda)^{\frac{n-k}{n}} \left(\frac{wd}{\lambda(n-k)e}\right)^{\frac{wd}{n}} e^{-\rho \frac{w}{n}} \right.$$

Set $x = w/k$, $\phi = \frac{n}{k} = \frac{1}{R}$

Above version of

$$\frac{\phi}{\phi}\ln b + \frac{x}{\phi}\ln\left(\frac{\phi e}{x}\right) + \left(\frac{\phi-1}{\phi}\right)\ln\cosh\lambda$$

$$+ \frac{xd}{\phi}\ln\left(\frac{d}{\lambda e(\phi-1)}\right) - \rho\frac{x}{\phi} < 0$$

Set λ s.t. $x = \frac{\lambda \tanh\lambda}{d}$:

$$\phi\ln b + \frac{\lambda\tanh\lambda}{d}\ln\left(\frac{\phi e d}{\lambda\tanh\lambda}\right) + (\phi-1)\ln\cosh\lambda$$

$$+ \lambda\tanh\lambda\ln\left(\frac{d\lambda\tanh\lambda}{d\lambda e(\phi-1)}\right) - \frac{\rho\lambda\tanh\lambda}{d} < 0$$

$$\Leftrightarrow \quad \rho > \frac{d}{\lambda\tanh\lambda}\left[\phi\ln b + \frac{\lambda\tanh\lambda}{d}\ln\left(\frac{\phi e d}{\lambda\tanh\lambda}\right)\right.$$

$$\left. + (\phi-1)\ln\cosh\lambda + \lambda\tanh\lambda\ln\left(\frac{\tanh\lambda}{e(\phi-1)}\right)\right]$$

$$\triangleq f(\lambda,\phi), \quad \text{where } \phi = \frac{1}{R}$$

$$b = 2^{\mathrm{Cap}(C)-R}$$

$$\boxed{\rho > \max_{\lambda>0} f(\lambda,\phi)}$$

{ we maximize over λ [illegible] d [illegible] a particular value [illegible] this bound [illegible] }

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$\operatorname{sgn}(n-k) = \begin{cases} +1 & \text{if } n \geq k \\ -1 & \text{if } n < k \end{cases} = \begin{cases} +1 & \text{if } R \leq 1 \\ -1 & \text{if } R > 1 \end{cases}$$

<u>2^{nd} approach, middle part of the sum:</u>

We use the expression in pg 13 of Pakzad's p

{ Replace k with $n-k$:
Pakzad's case: left-null space of G^T $(n \times k)$
Our case: left-null space of H^T $(n \times n-k)$ }

CAUTION: Not valid if $x, \phi < 0$!

$$\binom{n}{w} = \frac{\phi}{\sqrt{2\pi x(\phi-x)\phi(n-k)}} \left(\frac{\phi^{\phi}(\phi-x)^{x}}{x^{x}(\phi-x)^{\phi}} \right)^{n-k} (1+o($$

where $x = \dfrac{w}{n-k}$, $\phi = \dfrac{n}{n-k} = \dfrac{1}{1-R}$

$$b^n B_w e^{-\rho w} \leq b^n \, 2 \binom{n}{w} \cosh(\lambda)^{n-k} \left(\frac{wd}{\lambda e(n-k)} \right)^{wd} wd$$

$$= \frac{2\phi b^n}{\sqrt{2\pi x(\phi-x)\phi(n-k)}} \left(\frac{\phi^{\phi}(\phi-x)^{x}}{x^{x}(\phi-x)^{\phi}} \right)^{n-k} \cosh(\lambda)^{n-k} \left(\frac{wd}{\lambda e(n-k} \right.$$

$$wd \, e^{-\rho w} (1+o($$

Set $x = a\dfrac{\lambda \tanh \lambda}{d} \Rightarrow w = (n-k)\dfrac{\lambda \tanh \lambda}{d} a$, where $a =$ (see to

$a \geq$

$$b^n B_w e^{-\rho w} \leq \frac{2\phi wd \, b^n}{\sqrt{2\pi x(\phi-x)\phi(n-k)}} \left(\frac{\phi^{\phi}(d\phi - a\lambda \tanh\lambda)}{\cancel{d^{a\lambda\tanh\lambda/d}} \left(\frac{a\lambda\tanh\lambda}{\cancel{d}}\right)^{a\lambda\tanh\lambda/d}} \right.$$

$$\cdot \left. \frac{d^{\phi}}{(d\phi - a\lambda\tanh\lambda)^{\phi}} \cdot \cosh(\lambda) \right]^{n-k} \cdot \frac{\left[a\cancel{(n-k)}\cancel{\lambda}\tanh\lambda\right]^{a(n-k)\lambda}}{\left[\cancel{\lambda} e \cancel{(n-k)}\right]^{a(n-k)\lambda\tanh}}$$

$$\cdot \, e^{-\rho a(n-k)\frac{\lambda}{d}\tanh\lambda} \left(1+o(1)\right)$$

$$= \frac{2\phi wd}{\sqrt{2\pi x(\phi-x)\phi(n-k)}} \left[\frac{(\phi d)^{\phi} (d\phi - a\lambda\tanh\lambda)^{\frac{a\lambda\tanh\lambda}{d} - \phi}}{(a\lambda\tanh\lambda)^{\frac{a\lambda}{d}\tanh\lambda}} \cosh\lambda \cdot \left(\frac{a\tanh\lambda}{e}\right)^{a\lambda\tanh} \right.$$

$$\left. e^{-a\rho\frac{\lambda}{d}\tanh\lambda} \cdot b^{\frac{\phi}{\cancel{\phi-1}}} \right]^{n-k} (1+o($$

Note that $x = a\dfrac{\lambda\tanh\lambda}{d} = \dfrac{w}{n-k} \leq \dfrac{n}{n-k} = \phi = \dfrac{1}{1-R}$

Define

$$g(\phi,\lambda) = (\phi d)^{\phi} \, (d\phi - a\lambda \tanh\lambda)^{\frac{a\lambda\tanh\lambda}{d} - \phi} \cosh\lambda \left(\frac{a\tanh\lambda}{e}\right)^{a\lambda\tanh} \cdot e^{-Pa\frac{\lambda\tanh\lambda}{d}} \cdot b^{\frac{\phi}{\phi-1}} \cdot \frac{1}{(a\lambda\tanh\lambda)^{a\frac{\lambda}{d}\tanh}}$$

Need $g(\phi,\lambda) < 1$

$$\Leftrightarrow \quad e^{+Pa\frac{\lambda\tanh\lambda}{d}} > (\phi d)^{\phi} (d\phi - a\lambda\tanh\lambda)^{\frac{a\lambda\tanh\lambda}{d} - \phi} \cosh\lambda \cdot \left(\frac{a\tanh\lambda}{e}\right)^{a\lambda\tanh\lambda} \cdot b^{\frac{\phi}{\phi-1}} \cdot \frac{1}{(a\lambda\tanh\lambda)^{\frac{a}{d}\lambda\tanh}}$$

$$\Leftrightarrow \quad P > \frac{d}{a\lambda\tanh\lambda} \ln\left[\quad '' \quad \right]$$

$$\triangleq g'(\lambda,\phi)$$

$\phi = \frac{1}{1-R}$ is fixed

$$\boxed{\cancel{P} > \max_{\lambda} g'(\lambda,\phi)}$$

04/02/2010 MEETING WITH AMIN

[illegible] of edges of deg 2 → higher [illegible] well [illegible]
[illegible] in expansion of h[illegible]

Sason - Wi[illegible]

$$\begin{cases} ? & > D \\ 0 & \le D \end{cases} \quad [illegible]$$

$$\frac{I(x,z)}{n} = \frac{H(X) - H(X|Z)}{n} < Cap(C)(1-$$

$H(X|Z) \geq ?$