ALTERNATIVE: Use the facts given in Waday. 2008 arxiv submission:

Let H be of size $m \times n$.

A_w : # of vectors $\underline{x}$ of weight w such tha $H\underline{x} = \underline{0}$

length n

For H being drawn from the random linear code ensemble (each entry iid Bernoulli $\frac{1}{2}$) is known that

$$\mathbb{E}[A_w] = \binom{n}{w} 2^{-m}, \text{ for } w \geq 1$$

Gallag monogr

Check: $1 + \sum_{w=1}^{n} \mathbb{E}[A_w] = 1 + \sum_{w=1}^{n} \binom{n}{w} 2^{-m}$

$$= 1 + 2^{-m}\left[2^n - 1\right]$$

$$= 1 + 2^{n-m} - \frac{1}{2^m} > 2^{n-m}$$

could be possible

If H is full-rank, nulli is $n-m$. If H is rank < nullity is $> n-m$.

We set $m=k$ to analyze the right kernel of generator matrix (i.e., the dual code)

$$\sum_{w=0}^{n} \mathbb{E}[B_w](1-2p)^{2w} = 1 + \sum_{w=1}^{n} \binom{n}{w} 2^{-k} (1-2p)^{2w}$$

$$= 1 + 2^{-k}\left[1 + (1-2p)^2\right]^n$$

$$= 1 + 2^{-k}\left[1 + (1-2p)^2\right]^n = 1 + \left[2^{-R}\left(1 + (1-2p)^2\right)\right]^n$$

$$2^{-R}\left[1 + (1-2p)^2\right] < 1$$

$$\Leftrightarrow 2^R > \left[1 + (1-2p)^2\right]$$

$$\iff \quad R > \log_2 \left\{ 1 + (1-2p)^2 \right\}$$

$$\mathrm{Cap}\big(BSC(p)\big) = 1 - h(p)$$
$$= 1 + p\log_2(p) + (1-p)\log_2(1-p)$$

Computing $\dfrac{1-h(p)}{\log_2\left[1+(1-2p)^2\right]}$ gives us the same values that we obtain for the d-uniform ensemble with large d!

QUESTION: How much do we need to tradeoff in capacity, to push the above ratio to 1?

$$\Pr\left\{ I(X;Z) < n\,\mathrm{Cap}(\tau) - n\varepsilon \right\}$$
$$\leq n - \mathbb{E}[H(Z)] - n\varepsilon \qquad \left(\text{from the proof of Lemma 4}\right)$$
$$\leq \log_2\left(\sum_{w=0}^{n} \mathbb{E}[B_w](1-2p)^{2w} \right) - n\varepsilon$$
$$= \log_2\left(\frac{1}{2^{n\varepsilon}} \sum_{w=0}^{n} \mathbb{E}[B_w](1-2p)^{2w} \right)$$

For the iid random ensemble,

$$\Pr\left\{ I(X;Z) < n\,\mathrm{Cap}(\tau) - n\varepsilon \right\}$$
$$\leq \log_2\left\{ 1 + \left[2^{-R}\left(1+(1-2p)^2\right)\right]^n \right\} - n\varepsilon$$

$$1 + \left[2^{-R}\left(1+(1-2p)^2\right)\right]^n = 2^{n\varepsilon}$$
$$\triangleq 1 + \delta^n$$

$$2^{n\varepsilon} = 1 + \delta^n$$
$$\delta^n = \left(2^{n\varepsilon} - 1\right)$$
$$\delta = \left(2^{n\varepsilon} - 1\right)^{1/n}$$

$\mathbb{E}[f(x)] \le f(\mathbb{E}(x))$ $\quad \frac{1}{1+\delta^n} \cdot n\delta^{n}$

$$2^{-R}\left(1+(1-2p)^2\right) = \delta$$

$$R > \log_2\left\{\frac{1+(1-2p)^2}{\delta}\right\}$$

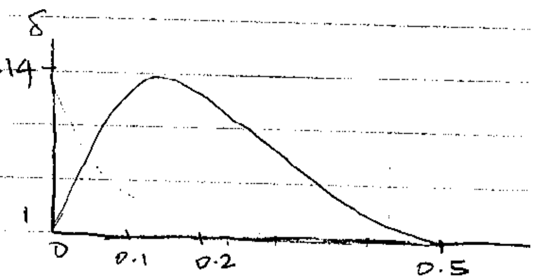
Equate : $\log_2\left\{\frac{1+(1-2p)^2}{\delta}\right\} = 1-h(p)$

$\delta = f(p)$, some $f(\cdot)$

Then obtain $\varepsilon = \frac{1}{n}\log_2(1+\delta^n) = \log_2(1+$

$$\frac{1+(1-2p)^2}{\delta} = 2^{1-h(p)}$$

$$\delta = \frac{1+(1-2p)^2}{2^{1-h(p)}}$$



$$\varepsilon = \frac{\log_2 e}{n}\left[\delta^n - \frac{\delta^{2n}}{2} + \frac{\delta^{3n}}{3} - \dots\right]$$

large n

$$\approx \frac{\delta^n}{n}\log_2 e$$

MACMULLAN / COLLINS CONTD. (From 2 pages back)

$$H(R) = (1-p)^n \log\frac{1}{(1-p)^n}\sum_{i=1}^{2^{n-\bar{k}}} W_{c,i}\left(\frac{p}{1-p}\right)$$

$$- (1-p)^n \sum_{i=1}^{2^{n-\bar{k}}} W_{c,i}\left(\frac{p}{1-p}\right)\log W_{c,i}\left(\frac{p}{1-p}\right)$$

$\bar{k} \rightarrow$ Average dimensionality of the random ensemble of c

$$\geq (1-p)^n\log\frac{1}{(1-p)^n}\sum_{i=1}^{2^{n-\bar{k}}} W_{c,i}\left(\frac{p}{1-p}\right) - (1-p)^n\log\left[\sum_{i=1}^{2^{n-\bar{k}}} W^2_{c,i}\left(\frac{p}{1-p}\right)\right]$$

CHECK: Can we use iterated expectation to average over $\bar{k}$ & $\mathbb{E}[W_{c,i}]$ separately?

$$\mathbb{E}\left(H(R)\right) \geq (1-p)^n \log \frac{1}{(1-p)^n} \sum_{i=1}^{2^{n-k}} \mathbb{E}\left[W_{c,i}\left(\frac{p}{1-p}\right)\right]$$
$$- (1-p)^n \log \left\{ \sum_{i=1}^{2^{n-\bar{k}}} \mathbb{E}\left[W_{c,i}^2 \left(\frac{p}{1-p}\right)\right]\right\}$$

5/07/10 CHOOSING AN APPROPRIATE BACKOFF FROM CAPACITY

Consider

$$\Pr\{ I(X;Z) < n(1-\delta)\, Cap(\mathcal{E}) \}$$
$$\leq n(1-\delta) \underbrace{Cap(\mathcal{E})}_{1-h(p)} - \mathbb{E}[H(Z)] + n\, h(p)$$
$$= n - \mathbb{E}[H(Z)] - n \underbrace{\delta(1-h(p))}_{\triangleq \epsilon}$$
$$\leq \log_2 \left\{ 1 + \left[2^{-R}\left(1 + (1-2p)^2\right)\right]^n \right\} - n\epsilon$$

Need to have

$$1 + \left[2^{-R}\left(1+(1-2p)^2\right)\right]^n = 2^{n\epsilon} \triangleq 1 + 2^{n\beta}$$
$$\Rightarrow \quad 2^{-R}\left(1 + (1-2p)^2\right) \leq 2^{\beta}$$

check: does the bound on prob. become -ve?

$$R \geq \log_2\left[1 + (1-2p)^2\right] - \beta$$

Would like the above threshold to be $1-h(p)$ (choose β such that this happens)

$$1-h(p) = \log_2\left[1+(1-2p)^2\right] - \frac{\log_2\left(2^{n\epsilon}-1\right)}{n}$$

$$\simeq \log_2$$

WORKING WITH THE VANISHING LOG BACKOFF

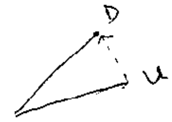
$$\Pr\{I(x;z) \le n\,Cap(\epsilon) - \log_2(1+\delta^n)\}$$

$$\le \log_2\left\{\frac{1}{1+\delta^n}\sum_{w=0}^{n}\mathbb{E}[B_w](1-2p)^{2w}\right\}$$

$$= \log_2\left\{\frac{1}{1+\delta^n} + \underbrace{\frac{1}{1+\delta^n}\sum_{w=1}^{n}\mathbb{E}[B_w](1-2p)}_{\delta'}\right.$$

As $n \to \infty$, determine the R for which $\delta' -$

$$\lim_{n\to\infty}\frac{1}{1+\delta^n} \to 1 \quad \text{for } \delta < 1$$



$D + x = u$

CLOSER LOOK AT THE ENTROPY BOUND:

Taylor series expansion of the entropy functio

$$H(D) = H(u) + \nabla^T (D - u)$$

$$+ \frac{1}{2}(D-u)^T \mathcal{H} (D-u)$$

$$\nabla = \nabla_D [H] \Big|_{D=u} = \begin{bmatrix} \frac{\partial H}{\partial D_1} \\ \vdots \\ \frac{\partial H}{\partial D_n} \end{bmatrix} \Bigg|_{D=u}$$

$$\mathcal{H} = \mathcal{H}_D [H] \Big|_{D=u} = \begin{pmatrix} \frac{\partial^2 H}{\partial D_1^2} & \cdots & \frac{\partial^2 H}{\partial D_1 \partial D_n} \\ \vdots & & \vdots \\ \frac{\partial^2 H}{\partial D_1 \partial D_n} & \cdots & \frac{\partial^2 H}{\partial D_n^2} \end{pmatrix} \Bigg|_{D=u}$$

$$H(x) = -\sum_{x=1}^{n} x_i \log_2 x_i$$

$$\frac{\partial H}{\partial x_j} = -\cancel{x_j} \cdot \left(\frac{1}{\cancel{x_j}}\right) - \log_2 x_j = -(1 + \log_2 x$$

Evaluating it at $x_j = \frac{1}{2^n}$,

$$\frac{\partial H}{\partial x_j}\Big|_{x_j = \frac{1}{2^n}} = -(1-n) = n-1$$

$$\nabla^T (D-u) = (n-1)\,[1\ 1 \ldots 1] \begin{bmatrix} D_1 - \frac{1}{2^n} \\ \vdots \\ D_{2^n} - \frac{1}{2^n} \end{bmatrix}$$

$$= (n-1)\left[\underbrace{\left(\sum_{i=1}^{2^n} D_i\right)}_{=1} - 1\right] = 0$$

$\geqslant n - .10$

bd: $\geqslant n - .26$

x

$\frac{1}{2}(1-x)$

$$\frac{\partial^2 H}{\partial x_i \partial x_j} = 0 \quad \forall \; i \neq j$$

$$\frac{\partial^2 H}{\partial x_i^2} = -\frac{1}{x_i}$$

$$\mathcal{H} = -2^n I_{2^n \times 2^n}$$

$$\frac{1}{2}(D-\mathcal{U})^T \mathcal{H} (D-\mathcal{U}) = -2^{n-1} \| D - \mathcal{U} \|^2$$

$$= -2^{n-1} \sum_{i=1}^{2^n} \| D_i - \tfrac{1}{2^n} \|^2 \quad \text{— (A)}$$

Compare with

$$-2^n \sum_{i=1}^{2^n} D_i^2 \quad \text{— (B)}$$

$$\text{(A)} : \quad \| D_i - \tfrac{1}{2^n} \|^2 = D_i^2 + \frac{1}{2^{2n}} - \frac{2D_i}{2^n}$$

$$\text{(A)} = -\frac{2^n}{2} \left[\sum_{i=1}^{2^n} D_i^2 + \cancel{2^n} \cdot \frac{1}{2^{\cancel{2}n}} - \frac{2}{2^n}(1) \right]$$

$$= -\frac{2^n}{2} \left[\sum_{i=1}^{2^n} D_i^2 - \frac{1}{2^n} \right]$$

$$= -2^{n-1} \sum_{i=1}^{2^n} D_i^2 + \frac{1}{2}$$

$$= \frac{1}{2} \left[1 - 2^n \sum_{i=1}^{2^n} D_i^2 \right]$$

From the proof of the main theorem in our paper

$$2^n \sum_{i=1}^{2^n} D_i^2 \equiv 2^n \sum_{u \in \mathbb{F}_2^n} P_u^2 = \sum_{v \in \mathbb{F}_2^n} (1 - 2q_v)^2$$

$\frac{1}{2}(x)^n$, $\log(1+x^n)$

$$Ⓐ = \frac{1}{2}\left[1 - \sum_{v \in \mathbb{F}_2^n} (1-2q_v)^2\right]$$

If $v \notin C^\perp$, $q_v = \frac{1}{2} \Rightarrow 1-2q_v = 0$

If $v \in C^\perp$, $q_v = \dfrac{1 - \prod_{i, v_i=1} (1-2p_i)}{2}$

$$Ⓐ = \frac{1}{2}\left[1 - \sum_{v \in C^\perp} \prod_{i, v_i=1} (1-2p_i)^2\right]$$

$$\equiv \frac{1}{2}\left[1 - \sum_{w=0}^{n} B_w (1-2p)^{2w}\right]$$

$$\therefore \quad n - \mathbb{E}[H(Z)] = \frac{1}{2}\left[\sum_{w=0}^{n} B_w (1-2p)^{2w} - \right.$$

The case of the iid random ensemble:

$$n - \mathbb{E}[H(Z)] = \frac{1}{2}\left[2^{-R}\left(1 + (1-2p)^2\right)\right]^n$$

HIGHER-ORDER TAYLOR-SERIES TERMS:

Only order 3 term that survives:

$$\frac{\partial^3 H}{\partial x_i^3} = \frac{1}{x_i^2} \quad ; \quad \left.\frac{\partial^3 H}{\partial x_i^3}\right|_{x_i = 1/2^n} = 2^{2n}$$

$$\sum_{i=1}^{2^n}\left[-\left(D_i - \tfrac{1}{2^n}\right)^2 \frac{2^n}{2!} + \left(D_i - \tfrac{1}{2^n}\right)^3 \frac{2^{2n}}{3!} - \left(D_i - \tfrac{1}{2^n}\right)^4 \frac{2^{3n}}{4!} + \cdots\right]$$

$$= \frac{1}{2^n} \sum_{j=1}^{\infty} \sum_{i=1}^{2^n} \frac{\left[2^n\left(D_i - \frac{1}{2^n}\right)\right]^j}{j!} (-1)^{j+1}$$

$$= \frac{1}{2^n} \sum_{i=1}^{2^n} \left[1 - e^{-2^n\left(D_i - \frac{1}{2^n}\right)}\right]$$

$$\left\{ e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \right\}$$

$$= 1 - \frac{1}{2^n} \sum_{i=1}^{2^n} \left(e^{1 - 2^n D_i} \right)$$

Taylor series:

$$T(x_1, \dots, x_d) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{(x_1-a_1)^{n_1} \cdots (x_d-a_d)^{n_d}}{n_1!\, n_2! \cdots n_d!} \left(\frac{\partial^{n_1+\cdots+n_d} f}{\partial x_1^{n_1} \cdots \partial x_d^{n_d}} \right)(a_1, \dots, a$$

$$H(x_1, \dots, x_{2^n}) = H\left(\tfrac{1}{2^n}, \dots, \tfrac{1}{2^n}\right) + \sum_{i=1}^{\infty} \sum_{j=1}^{2^n} \frac{(x_j - \frac{1}{2^n})^i}{i!} \left(\frac{\partial^i H}{\partial x_j^i} \right) \left(\tfrac{1}{2^n}, \dots,\right.$$

any "mixed" partial derivative is zero

$i=1$ term reduces to ze

$$H(x_1, \dots, x_{2^n}) = - \sum_{i=1}^{2^n} x_i \log x_i$$

$$\frac{\partial H}{\partial x_j} = -\cancel{x_j}\left(\frac{1}{\cancel{x_j}}\right) - \log x_j = -1 - \log x_j$$

$$\frac{\partial^2 H}{\partial x_j^2} = -\frac{1}{x_j} \; ; \quad \frac{\partial^3 H}{\partial x_j^3} = \frac{1}{x_j^2} \; ; \quad \frac{\partial^4 H}{\partial x_j^4} = (-2) \cdot \frac{1}{x}$$

$$\frac{\partial^5 H}{\partial x_j^5} = (-2)(-3) \frac{1}{x_j^4}$$

$$\frac{\partial^i H}{\partial x_j^i} = (-1)^{i+1} \frac{(i-2)!}{x_j^{(i-1)}} \, , \quad i \geq 2$$

$$H(x_1, \dots, x_{2^n}) = n + \sum_{i=2}^{\infty} \sum_{j=1}^{2^n} \frac{(x_j - \frac{1}{2^n})^i}{i!} (-1)^{i+1} (i-2)! \cdot (2^n)^{i-}$$

$$= n + \frac{1}{2^n} \sum_{i=2}^{\infty} \sum_{j=1}^{2^n} (-1)^{i+1} \frac{\left[2^n (x_j - \frac{1}{2^n}) \right]^i}{i(i-1)} \quad — \quad (1)$$

$f(E(x)) \geq E[f(x)]$
$-f(E(x)) \leq -E[f(x)]$

MAPLE says

$$\sum_{i=2}^{\infty} (-1)^{i+1} \frac{x^i}{i(i-1)} = (-1-x)\ln(x+1) + x$$

$$H(x_1, \ldots, x_{2^n}) = n + \frac{1}{2^n} \sum_{j=1}^{2^n} \Big[(2^n x_j - 1) + (\cancel{1} - 2^n x_j \cancel{+1}) \ln(2^n x_j$$

$$= n + \frac{1}{2^n} \sum_{j=1}^{2^n} \left[2^n x_j - 1 - 2^n x_j \ln(2^n x_j) \right]$$

$$= n + \cancel{\frac{1}{2^n}(2^n)(1)} - \cancel{\frac{1}{2^n}(2^n)} - \sum_{j=1}^{2^n} x_j \ln(2^n$$

NEED $H(x_1, \ldots, x_{2^n}) \geq n - \log\left(\sum_{i=1}^{2^n} x_i^2\right)$

Suppose we ignore the "$i-1$" in the denominator

$$n + \frac{1}{2^n} \sum_{j=1}^{2^n} \left[\sum_{i=2}^{\infty} (-1)^{i+1} \frac{[2^n x_j - 1]^i}{i} + \frac{[2^n x_j - 1]}{1} - [$$

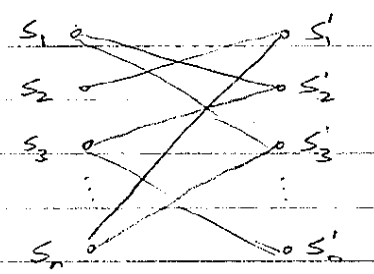
$$= n + \frac{1}{2^n} \sum_{j=1}^{2^n} \left[\ln(2^n x_j) - 2^n x_j + 1 \right]$$

$$n - \log\left(\sum_{i=1}^{2^n} x_i^2\right) = n - \left(\sum_i x_i^2 - 1\right) + \frac{1}{2}\left(\sum_i x_i^2 - 1\right)^2 - \frac{1}{3}\left(\sum_i x_i^2 - 1\right)^3 + \cdots$$

$$= n + \sum_{i=1}^{\infty} (-1)^i \frac{\left(\sum_{j=0}^{2^n} x_j^2 - 1\right)^i}{i}$$

23/07/10 **EVOLUTION OF PROBABILITIES IN HOPFIELD N/WS**

Assume for simplicity that $w_{ij} = w_{ji}$ (symmetric)



$$P(S_i' = 1) = g(h_i)$$
$$= g\left(\sum_j w_{ij} S_j\right)$$
$$= \frac{1}{1 + \exp\left(-2\beta \sum_j w_{ij} S_j\right)}$$

26/07/10 **TAYLOR-SERIES FOR BOUNDS ON ENTROPY**:

$$f(\underline{x}) = -\log\left(\sum_{i=1}^{2^n} x_i^2\right)$$

$\left\{ \text{Know that } -H(D) \le -f(D) \Rightarrow H(D) \ge f(D) \right\}$

$$\frac{\partial H}{\partial x_j} = -\frac{1}{\sum_{i=1}^{2^n} x_i^2} \cdot (2x_j) \quad ; \quad \left.\frac{\partial H}{\partial x_j}\right|_{x_i = \frac{1}{2^n} \forall i} = -\frac{2}{2^n} \cdot \frac{1}{\sum_{i=1}^{2^n} \frac{1}{4^n}} = -\frac{2}{2^n} \cdot 2^n = -2$$

$$\frac{\partial^2 H}{\partial x_j^2} = (-2)\left[x_j \left(\frac{-1}{\left(\sum_{i=1}^{2^n} x_i^2\right)^2}\right)(2x_j) + \frac{1}{\sum_{i=1}^{2^n} x_i^2} \right]$$

$$\nabla^T (D - \mathcal{U}) = (-2)\,[1\ 1 \cdots 1] \begin{bmatrix} D_1 - \frac{1}{2^n} \\ \vdots \\ D_{2^n} - \frac{1}{2^n} \end{bmatrix} = 0$$

Go back 1 page, to ①:

What is the coeff. of x_k^2 in ①?

$$H(x_1, \dots, x_{2^n}) = n + \sum_{j=1}^{2^n} \sum_{i=1}^{\infty} a_{ij} x_j^i$$

Need: a_{2k}

$$\frac{\partial^2 H}{\partial x_k^2} = \left[2 a_{2k} + \text{terms involving } x_k \right]$$

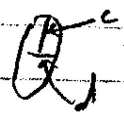
$$\frac{1}{2} \cdot \frac{\partial^2 H}{\partial x_k^2} \Big|_{x_k = 0} = a_{2k}$$

$H(x_1, \dots, x_m)$; $\sum x_i \leq 1$

$u = \left(\frac{1}{m+1}, \dots, \frac{1}{m+1} \right)$

entropy of $\underline{x}$, expanded arou

$$d \|X\|_2^2 \leq \underbrace{\log(\,)}_{\cancel{H(u)}} - \underbrace{H(\underline{X} - u)} \leq c \cdot \|X\|_2^2$$



$$1 - \frac{(x - 1/2)^2}{\ln 2} - \dots \leq 1 - \frac{(x \cdot}{\ln}$$

$$\overset{?}{\geq} 1 - H(x - 1/2) \geq \frac{(x - 1/2)^2}{\ln 2}$$

9/8/10 <u>TAYLOR EXPANSIONS FOR $h(x)$</u>:

$$h(x) = -x \log x - (1-x) \log (1-x)$$

$$= 1 - \frac{2}{\ln 2}\left(x - \frac{1}{2}\right)^2 - \frac{4}{3 \ln 2}\left(x - \frac{1}{2}\right)^4 - \frac{32}{15 \ln 2}\left(x - \right.$$

$$+ o\left(\left(x - \frac{1}{2}\right)^8\right)$$