Summary of Results on MI Bounds for LDPC/LDGM Codes Obtained from Statistical Mechanics

Raj & Hesam

Consider the transmission of LDGM or LDPC codes over a BIMSC, and let X, Y and Z denote respectively the vector of information symbols, the codeword and the output of the channel respectively. For this scenario, several results have reported over the past few years regarding bounds on the mutual information $I(X;Z)$ results obtained using statistical mechanics (in particular, the replica method of spin-glass theory). Among the first of these results was the work by Montanari [1]. Following this, several papers have tried to improve the results of [1]. We summarize briefly the results from this series of works in this report.

Tight bounds for LDPC and LDGM codes under MAP decoding - Montanari [1]

This paper considers the transmission of either LDPC or LDGM codes over arbitrary BIMSCs. In order to present the main result of this paper, consider the following definition.

Definition 1: Fix a degree sequence pair (Λ, P), and let ρ_k be the edge perspective right degree distribution $\rho_k = P_k/P'(1)$. Let $V_1, V_2, ... =_d V$ be a family of i.i.d. symmetric random variables, k an integer with distribution ρ_k, J a symmetric random variable distributed as the log-likelihoods $\{J_a\}$ of the parity check bits, and

$$U^V = \tanh^{-1}\left[\tanh J \prod_{i=1}^{k-1} \tanh V_i\right].$$

The random variables U, V are said to be admissible if they are independent, symmetric and $U =_d U^V$. For any couple of admissible random variables U, V, we define the associated trial entropy as follows

$$\phi_V(\Lambda, P) = -\Lambda'(1)\mathbb{E}_{u,v}\log_2\left[\sum_x P_u(x)P_v(x)\right] + \mathbb{E}_l\mathbb{E}_y\mathbb{E}_{\{u_i\}}\log_2\left[\sum_x \frac{Q_{\mathcal{V}}(y|x)}{Q_{\mathcal{V}}(y|0)}\prod_{i=1}^{l}P_{u_i}(x)\right] + \\ + \frac{\Lambda'(1)}{P'(1)}\mathbb{E}_k\mathbb{E}_{\hat{y}}\mathbb{E}_{\{v_i\}}\log_2\left[\sum_{x_1\dots x_k}\frac{Q_{\mathcal{C}}(\hat{y}|x_1\oplus\cdots\oplus x_k)}{Q_{\mathcal{C}}(\hat{y}|0)}\prod_{i=1}^{k}P_{v_i}(x_i)\right], \quad (1)$$

where l and k are two integer random variables with distribution (respectively) Λ_l and P_k. Hereafter we shall drop the reference to the degree distributions in $\phi_V(\Lambda, P)$ whenever this is clear from the context.

The main result of the paper is the following.

Theorem 1: Let $P(x)$ be a polynomial with non-negative coefficients such that $P(0) = P'(0) = 0$, and assume that $P(x)$ is convex for $x \in [-x_0, x_0]$. Let

$$h_n = \frac{1}{n}\mathbb{E}\left[H(X|Z)\right]$$

be the expected conditional entropy per bit for either of the standard ensembles LDGM(n, Λ, P) or LDPC(n, Λ, P). If $x_0 > e$, then

$$\lim\inf_{n\to\infty} h_n \geq \sup_V \phi_V(\Lambda, P).$$

In a nutshell, the main result is the following:

- It results in an upper bound for the mutual information
- It is valid for LDPC or LDGM ensembles where the right degree distribution is a convex polynomial (in particular, if the check degree is constant, this means that it has to be even), over any BIMSC
- The result is obtained through techniques in statistical physics that are not entirely rigorous, which is why the paper can only claim a bound and not an exact expression for the entropy. These methods from statistical physics are believed (conjectured) to be exact, but haven't been proven in the general case. We will look at a few papers subsequently, that have shown that the bound is tight in certain restrictive scenarios.
- Computing these bounds exactly is a difficult problem. Suboptimal bounds are obtained in [1] by evaluating (1) only at particular distributions (Monte Carlo). Also, [1] optimizes these suboptimal bounds through message passing techniques.
- Simulation results are presented for regular LDPC codes over the BEC and BSC channels. For the BSC, results are plotted in terms of the maximum value of transition probability such that the code is capacity achieving. The upper bound on MI that is obtained in this paper is shown to be tighter than Gallager's upper bound.

Exact solution for the conditional entropy of Poissonian LDPC codes over the Binary Erasure Channel Channel - Korada, Kudekar and Macris

In this paper, it is shown that the bound in [1] is exact for a particular ensemble of Poisson LDPC codes (where the number of check nodes is a Poisson RV, and these are connected uniformly at random to n variable nodes), over the BEC.

Sharp Bounds for Optimal Decoding of Low-Density Parity-Check Codes - Kudekar and Macris:

In this paper, the convexity requirement for $P(x)$ imposed in [1] is dropped in the cases of the BEC, binary input additive white Gaussian noise channel (BIAWGNC) with any noise level, and in the case of general BMS channels in a high noise regime. In other words, they prove the lower bound on the conditional entropy for any standard regular (so odd degrees are allowed) or irregular code ensemble. The bound is conjectured to be tight since it matches the replica solution.

Decay of Correlations in Low Density Parity Check Codes: Low Noise Regime - Kudekar and Macris:

This paper proves that the replica solution obtained in [1] is the exact expression for the input-output entropy for a particular class of LDPC codes (containing a fraction of degree one variable nodes, for example, the poisson ensemble) on the BIAWGN channel.

The Generalized Area Theorem and Some of its Consequences - Measson, Montanari, Richardson and Urbanke:

In this paper, the bound derived in [1] is rephrased in terms of the GEXIT (generalized EXIT) function, for a family of channels that can be ordered in terms of physical degradation. It is shown that the convexity requirement on the $P(x)$ in [1] is not necessary.

References

[1] A. Montanari, ”Tight bounds for LDPC and LDGM codes under MAP decoding,” *IEEE Trans. Inform. Theory*, Vol. 51, No. 9, pp. 3221-3246, Sep. 2005.