Exercise 1. In all this exercise, we work with a bipartite graph $G = (V, E)$.

1. Show that the following in an exact LP-relaxation (that is, it always has an integral optimal solution) for the maximum weight matching in G:

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_e x_e w_e \\ \text{subject to} \quad & \textstyle\sum_{e, v \in e} x_e \leq 1 \quad \forall v \in V \\ & x_e \geq 0 \quad \forall e \in E. \end{aligned}$$

   Hint: Show that the involved matrix is totally unimodular. For that, show that any square submatrice has either a determinant of 0 or contains a column with only one 1 (allowing an induction). Note that the fact that G is bipartite is essential for this proof and this is false for a general graph.

2. Obtain the dual of this LP problem, and show that in the case of all the w_e being one, it is an exact LP-relaxation for the minimum vertex cover problem.

3. Using the previous result, what "min-max" kind of theorem can you infer ? (This is known as the König-Ergeváry Theorem)

Exercise 2. In the previous exercise we saw an exact LP-relaxation for the maximal weight matching in a bipartite graph but not in a general graph. However this linear program can be used to construct a primal-dual approximation algorithm for this problem. Find an approximation algorithm for this problem and use the complementary slackness condition to prove that its approximation factor is 2.

Exercise 3. Construct a primal-dual algorithm for the $s-t$ shortest path problem (you can use the LP you derived in the last exercise sheet). How can you relate your algorithm to Dijkstra's algorithm?

Remark: Recall Dijkstra's algorithm:

DIJKSTRA(*Vertex set* V, *Edge set* E, *Cost function* c, *Source node* s, *Destination node* t)

$W \leftarrow \{\}$
$d[s] \leftarrow 0$
$d[p] \leftarrow \infty \quad \forall p \in V - \{s\}$
while $W \neq V$
  do $v \leftarrow \text{argmin}\{d[x] : x \notin W\}$
    $W \leftarrow W \cup \{v\}$
    $d[x] \leftarrow \min\{d[x], d[v] + c_{vx}\} \quad \forall x \in V - W$
return $d[t]$