Solution.

Exercise 1.

- It is easy to check that $(1.5, 0.5, 0)$ is a feasible solution for this linear program giving a value of 1.5 for the objective function.
- Let consider the dual program.

$$\begin{aligned} \text{minimize} \quad & y_1 + y_2 \\ \text{subject to} \quad & y_1 \geq 1 \\ & -y_1 + 2y_2 \geq 0 \\ & -y_2 \geq -2 \\ & y_1, y_2 \geq 0 \; . \end{aligned}$$

- It is easy to check that $(1, 0.5)$ is a valid solution for this problem that gives a value of 1.5 for the objective function.
- Applying the Duality theorem for linear programs, we see that both these solutions are optimal.

Exercise 2.

1. We can introduce a new variable y (representing an upper bound on all the $x_i, i \in I$) and rewrite the program as

$$\begin{aligned} \text{minimize} \quad & y \\ \text{subject to} \quad & \mathcal{A}x = b, \\ & y - x_i \geq 0 \quad \forall i \in I \\ & x, y \geq 0, \end{aligned}$$

   Notice that the condition $\mathcal{A}x = b$ is not in a standard form but can be rewritten as such by doubling the number of constraints: $\mathcal{A}x \geq b$ and $-\mathcal{A}x \geq -b$.

2. Here we can directly apply the trick of question 1 and introduce a variable t such that, for each point i, t is greater or equal to the distance of this point to the line $ax + b$. We also need two variables a and b that describe the line. An "almost linear" program for this problem is then:

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & t - (ax_i + b - y_i) \geq 0 \quad \forall i \\ & t - (y_i - ax_i + b) \geq 0 \quad \forall i \\ & t \geq 0 \end{aligned}$$

   Notice the two conditions for each i to ensure that t is greater than or equal to $|ax_i + b - y_i|$. The only problem left is that a and b have no reason to be positive, so we need to replace each of them by $a = a^+ - a^-$ and $b = b^+ - b^-$ with a^+, a^-, b^+, b^- being 4 positive variables.

Exercise 3. The linear programming formulation of this problem is really close to the one of the max-flow problem. Let $G = (V, E)$ a directed graph, and $c : E \to \mathcal{R}$ a cost function on the edges (their weight). We will use one variable x_e per edges. The problem can be formulated as

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_e x_e c(e) \\ \text{subject to} \quad & \forall v \in V \quad \textstyle\sum_{e=(v,u)} x_e - \sum_{e=(u,v)} x_e \geq b_v \\ & \forall e \in E \quad x_e \geq 0 \end{aligned}$$

where the b_v are all 0 except $b_s = 1$ and $b_t = -1$. It is easy to see that a solution where x_e is 1 on the edges that form a shortest path from s to t and 0 elsewhere is a feasible solution for this problem. Looking at the dual, we will see that this is an optimal solution.

The dual program of this program is given by

$$\begin{aligned} \text{maximize} \quad & y_s - y_t \\ \text{subject to} \quad & \forall e = (u, v) \in E \quad y_u - y_v \leq c(e) \\ & \forall v \in V \quad y_v \geq 0 \end{aligned}$$

Suppose we assign to y_v the length of the shortest path from v to t. Then, if $e = (u, v) \in E$ for sure $d(u, t) \leq d(v, t) + c(e)$. Moreover, the value of the objective function is the length of the shortest path from s to t and is the same as the one we obtained for the primal problem. Thus, we proved than an optimal solution of the primal linear program gives the length of the shortest path from s to t.