**Exercise 1.** Consider the semidefinite program corresponding to a relaxation of MAX-CUT as seen in the class. Show that the optimal value of this program is at least $T/2$, where $T$ is the sum of the weights of the edges.

**solution :** The semidefinite program is the following :

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2} \sum_{1 \leq i < j \leq n} w_{ij}(1 - y_{ij}) \\ \text{subject to} \quad & y_{ii} = 1 \quad \forall i \in \{1, \ldots, n\} \\ & Y \geq 0 \\ & Y \in M_n \end{aligned}$$

It is easy to see that if we take for Y the identity matrix, then all the condition are satisfied, moreover the value of the objective function for this choice is exactly $T/2$.

**Exercise 2.** Consider the problem MAX-CUT with the additional constraint that some specified pair of vertices are on the same/opposite side of the cut. Give a strict quadratic program and vector program relaxation for this problem. Show how the approximation algorithm seen in class can be adapted to this problem. What is its approximation factor ?

**solution :** The additional constraints are of the form $x_i x_j = \pm 1$ for some $i \neq j$. The addition of theses constraints leaves the quadratic program strict, and the new problem can still be expressed as a vector program and hence we can use exactly the same approximation algorithm. The approximated solution will satisfy the constraints because whatever random vector $r$ we take, if $x_i x_j = \pm 1$, their scalar product with $r$ will either be the same or the opposite. Hence, the approximate solution is compatible with the specified pairs of vertices. The approximation factor does not change and is still $1/2$.

**Exercise 3.** Give a quadratic program for MAX-2SAT. Using an extra variable, make this program strict (that is transform it so that there are no terms of degree 1) and relax it to a vector program.

**solution :** Given a 2-SAT instance with $n$ variables $x_i$ and some number of clauses $C$ each containing 2 literals, the problem is to find an assignment of the variables that makes the maximal number of clause true. We can transform this problem to a strict quadratic program as follow. We introduce one variable $y_i \in \{-1, 1\}$ per initial Boolean variable $x_i$. Let us define a value $v(C)$ of a clause $C$ to be $1$ if the clause is satisfied and $0$ otherwise. We adopt the convention

$$v(x_i) = (1 + y_i)/2 \quad \text{and} \quad v(\overline{x_i}) = (1 - y_i)/2$$

Hence for a clause $x_i \vee x_j$ we have

$$v(x_i \vee x_j) = 1 - v(\overline{x_i})v(\overline{x_j}) \ .$$

The problem can thus be written as

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{i=1}^n a_i y_i + \sum_{1 \leq i < j \leq n} b_{ij} y_i y_j \\ \text{subject to} \quad & y_i^2 = 1 \quad i \in \{1, \ldots, n\} \end{aligned}$$

with an appropriate choice of the $a_i$'s and $b'_{ij}s$.

However, this program is not strictly quadratic. To transform it, we can introduce a new variable $y_0$, and change the convention $v(x_i) = (1 + y_i)/2$ by $v(x_i) = (1 + y_i y_0)/2$. The quadratic program becomes

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{i=1}^n a_i y_0 y_i + \sum_{1 \leq i < j \leq n} b_{ij} y_i y_j \\ \text{subject to} \quad & y_i^2 = 1 \quad i \in \{0, \ldots, n\} \end{aligned}$$