Exercise 1.1.

(a) Suppose that $x \in f(A \cup B)$. Then, there is $y \in A \cup B$ such that $f(y) = x$. If $y \in A$, then $x \in f(A)$, and if $y \in B$, then $x \in f(B)$. So, $x \in f(A) \cup f(B)$. Conversely, suppose that $x \in f(A) \cup f(B)$. If $x \in f(A)$, then there is $y \in A$ such that $x = f(y)$. Similarly, if $x \in f(B)$, then there is $y \in B$ such that $x = f(y)$. Hence, $x \in f(A \cup B)$.

(b) Suppose that $x \in f(A \cap B)$. Then there is $y \in A \cap B$ such that $x = f(y)$. Hence, $x \in f(A)$ and $x \in f(B)$, which shows that $x \in f(A) \cap f(B)$.

(c) Suppose that $x \in f^{-1}(C \cup D)$. Then $f(x) =: y \in C \cup D$. If $y \in C$, then $f(x) \in C$, so that $x \in f^{-1}(C)$. Similarly, if $y \in D$, then $x \in f^{-1}(D)$. Hence $x \in f^{-1}(C) \cup f^{-1}(D)$. This shows that $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$. Conversely, suppose that $x \in f^{-1}(C) \cup f^{-1}(D)$. If $x \in f^{-1}(C)$, then $f(x) =: y \in C$. Similarly, if $x \in f^{-1}(D)$, then $f(x) \in D$. Hence, $f(x) \in C \cup D$, so that $x \in f^{-1}(C \cup D)$. This shows that $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}(C \cup D)$, and hence equality of the two sets is proved.

(d) Let $x \in f^{-1}(C \cap D)$. Then $f(x) =: y \in C \cap D$, and hence $y \in C$ and $y \in D$. Therefore, $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$ and hence $x \in f^{-1}(C) \cap f^{-1}(D)$. Conversely, suppose that $x \in f^{-1}(C) \cap f^{-1}(D)$. Then $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$, so $f(x) \in C \cap D$. It follows that $x \in f^{-1}(C \cap D)$.

Exercise 1.2.

(a) Let $f\colon X \to P(X)$ be defined by $f(x) := \{x\}$. This mapping is clearly injective.

(b) Let $g \in X \to P(X)$ be a mapping, and suppose that g is surjective. Consider $S := \{x \mid x \notin g(x)\}$. Since g is surjective, there exists $z \in X$ such that $S = g(z)$. By definition of S, we have that $z \notin S$: if $z \in S$, then $z \notin g(z) = S$. On the other hand, since $z \notin S$, it follows that $z \in g(z) = S$, a contradiction.

Exercise 1.3.

(a) Since $A - B = A \cap B^c$, we can use the De Morgan laws to see that

$$(A - B) \cup (B - A) = (A \cap B^c) \cup (B \cap A^c) = (A \cup B) \cap (A^c \cup B^c).$$

(b) Since $A\Delta B = A^c$, it follows by part (a) that $A \cup B \supseteq A^c$, so that $B \supseteq A^c$ since $A \cap A^c = \emptyset$. Therefore, $A \cup B = X$, and hence $A^c = A\Delta B = A^c \cup B^c$, so $B^c \subseteq A^c$, so $B \supseteq A$. It follows that $B \supseteq A \cup A^c = X$, so $B = X$.

(c) Follows from the definition.

(d) Let $\oplus$ denote the logical operation of XOR: if p and q are boolean variables, then $p \oplus q$ is true iff either p is true, or q is true. Then the assertion $x \in A\Delta B$ is equivalent to $(x \in A) \oplus (x \in B)$, by the definition of Δ. Hence $x \in (A\Delta B)\Delta C$ is the assertion $((x \in A) \oplus (x \in B)) \oplus (x \in C)$. Associativity now follows from the associativity of boolean formulas.

(e) $P(X)$ is closed under Δ, since this operation produces a subset of X; the operation is associative according to (d). The element $\emptyset$ is the neutral element with respect to Δ, since $A\Delta\emptyset = (A - \emptyset) \cup (\emptyset - A) = A \cup \emptyset = A$. The inverse of A with respect to this operation is A itself, since $(A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset$.

(f) We know by (a) that $A\Delta B = (A\cup B)\cap(A^c\cup B^c) = (A\cup B)-(A\cap B)$, since $C-D = C\cap D^c$, and $(A^c \cup B^c)^c = A \cap B$. Therefore, since $A \cap B \subseteq A \cup B$, $|A\Delta B| = |A \cup B| - |A \cap B|$, and by the principle of inclusion and exclusion, we have $|A \cup B| = |A| + |B| - |A \cap B|$, which proves the assertion.

Exercise 1.4. First we show that for all $x \in X$ there is $y \in Y$ such that $(x, y) \in F$. Let $S = \phi(\{x\})$. Then, since $\gamma(S) = \{y\}$ by assumption, we see that $S \neq \emptyset$. Since the conditions are symmetric in X and Y, we see that for every y there is an x such that $(x, y) \in F$. Hence, if F is the graph of a function, then this function is surjective. Next, we show that for every x, there is exactly one y such that $(x, y) \in F$. Suppose that $\phi(\{x\}) = S$. Then, since $\gamma(S) = \{x\}$, we have for all $y \in S$ that $\gamma(\{y\}) = \{x\}$. Hence, $\phi(\{x\}) = \phi(\gamma(\{y\})) = \{y\}$, hence there is only one y such that $(x, y) \in F$. Hence, F is the graph of a function. Since the conditions are symmetric in x and y, we see that F is also injective.

Exercise 1.5. The given numbers are possible. Let M denote the set of married students, H denote the set of male students, and V denote the set of students of age at least 21. Assume that the students are numbered from 1 to 10000. Then we can have the following setup:

$$\begin{aligned}
M &= \{1, \ldots, 2521\}\\
H &= \{607, 608, \ldots, 2521, 2522, \ldots, 7077\}\\
V &= \{36, \ldots, 606\} \cup \{607, 608, \ldots, 1908\} \cup \{2522, 2523, \ldots, 3763\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
|M| &= 2521\\
|H| &= 7077 - 606 = 6471\\
|V| &= 606 - 35 + 1908 - 606 + 3763 - 2521 = 3115\\
|H \cap M| &= |\{607, 608, \ldots, 2521\}| = 2521 - 606 = 1915\\
|M \cap V| &= |\{36, \ldots, 606\} \cup \{607, 608, \ldots, 1908\}| = 606 - 35 + 1908 - 606 = 1873\\
|H \cap M \cap V| &= |\{607, 608, \ldots, 1908\}| = 1908 - 606 = 1302.
\end{aligned}$$

These numbers are identical with the requirements given in the problem statement.