Exercise 12.1.

1. The series is $1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n = 1/(1-x)$.
2. Let $A = \sum_{n=0}^{\infty} x^n = 1/(1-x)$. Then

$$\partial(A) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n\geq 0} (n+1)x^n,$$

so the sequence is the sequence of positive integers.

3. Since $\partial(1/(1-x)^2)/2 = 1/(1-x)^3$, we have

$$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=0}^{\infty} n(n-1)x^{n-2} = \sum_{n\geq 0}\binom{n+2}{2}x^n.$$

So, for $n \geq 0$ the n-th term of the sequence is $\binom{n+2}{2}$, and the sequence is $1, 3, 6, \ldots$.

4. We have

$$\sum_{n\geq 0}(n+1)^2 x^n = \sum_{n\geq 0}(n+1)(n+2)x^n - \sum_{n\geq 0}(n+1)x^n.$$

By part 3 of this exercise, the first summand equals $2/(1-x)^3$, and by part 2 of this exercise the second summand equals $1/(1-x)^2$. So

$$\sum_{n\geq 0}(n+1)^2 x^n = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} = \frac{1+x}{(1-x)^3}.$$

Exercise 12.2.

1. We know that for $n \geq 1$: $a_{n+1} - 2a_n - 1 = 0$, so

$$\sum_{n\geq 0} a_{n+1}x^{n+1} - 2\sum_{n\geq 0} a_n x^{n+1} - \sum_{n\geq 0} x^{n+1} = 0.$$

Let $A := \sum_{n\geq 0} a_n x^n$. Then the first term in the above equation is $A - a_0 = A - 1$, the second term is $2xA$, and the third term is $x/(1-x)$. It follows that

$$A(1-2x) - \frac{1}{1-x} = 0 \implies A = \frac{1}{(1-x)(1-2x)} = \frac{2}{1-2x} - \frac{1}{1-x}.$$

Hence $A = \sum_{n\geq 0} 2^{n+1}x^n - \sum_{n\geq 0} x^n$, so $a_n = 2^{n+1} - 1$.

2. We know that for $n \geq 1$: $a_{n+1} - a_n - 2^n = 0$. Therefore

$$\sum_{n\geq 0} a_{n+1}x^{n+1} - \sum_{n\geq 0} a_n x^{n+1} - \sum_{n\geq 0} 2^n x^{n+1} = 0.$$

Let $A := \sum_{n\geq 0} a_n x^n$. Then the first term in the above equation is $A - a_0 = A - 1$, the second term is xA, and the third term is $x/(1-2x)$. It follows that

$$A(1-x) - \frac{x}{1-2x} - 1 = 0 \implies A(1-x) = \frac{1-x}{1-2x} \implies A = \frac{1}{1-2x}.$$

Hence $A = \sum_{n\geq 0} 2^n x^n$, so $a_n = 2^n$.

Exercise 12.3.

1. We know that for $n \geq 2$ we have $P_n - 2P_{n-1} - P_{n-2} = 0$. Therefore,

$$\sum_{n\geq 2} P_n x^n - 2\sum_{n\geq 2} P_{n-1} x^n - \sum_{n\geq 2} P_{n-2} x^n = 0.$$

Let $P = \sum_{n\geq 0} P_n x^n$. Then the first term equals $P - P_0 - P_1 x$, the second term is $x(P - P_0)$, and the third term is $x^2 P$. Since $P_0 = 0$ and $P_1 = 1$, we have

$$P - x - 2xP - x^2 P = 0 \implies P = \frac{x}{1-2x-x^2}.$$

This is the generating function for the numbers P_n.

2. By using Proposition 6.4(2), we find that

$$\begin{aligned}
\sum_{n\geq 0} P_n x^n &= \frac{x}{1-2x-x^2} \\
&= \frac{1}{2\sqrt{2}}\left(\frac{1}{1-x(1+\sqrt{2})} - \frac{1}{1-x(1-\sqrt{2})}\right) \\
&= \frac{1}{2\sqrt{2}}\sum_{n\geq 0}((1+\sqrt{2})^n - (1-\sqrt{2})^n)x^n.
\end{aligned}$$

Hence, $P_n = \frac{1}{2\sqrt{2}}((1+\sqrt{2})^n - (1-\sqrt{2})^n)$ for $n \geq 0$.

3. We have

$$\frac{P_{n+1}}{P_n} = \frac{(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}}{(1+\sqrt{2})^n - (1-\sqrt{2})^n} = \frac{1 - \left(\dfrac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{n+1}}{\dfrac{1}{1+\sqrt{2}} - \dfrac{1}{1-\sqrt{2}}\left(\dfrac{1-\sqrt{2}}{1+\sqrt{2}}\right)^{n+1}}.$$

Since $|(1-\sqrt{2})/(1+\sqrt{2})| < 1$, it follows that $P_{n+1}/P_n \to 1+\sqrt{2}$ as $n \to \infty$.

4. Since $(P_{n+1} - P_n)/P_n \to \sqrt{2}$ as $n \to \infty$, we have a quick algorithm for approximating $\sqrt{2}$ by fractions: we calculate the P_n, and apply the above formula. Below is a list of

these fractions, and their distance to $\sqrt{2}$:

n	P_n	$(P_n - P_{n-1})/P_{n-1}$	$(P_n - P_{n-1})/P_{n-1} - \sqrt{2}$
2	2	1	-0.4142135623
3	5	$\frac{3}{2}$	0.0857864376
4	12	$\frac{7}{5}$	-0.0142135623
5	29	$\frac{17}{12}$	0.0024531042
6	70	$\frac{41}{29}$	-0.0004204589
7	169	$\frac{99}{70}$	0.0000721519
8	408	$\frac{239}{169}$	-0.0000123789
9	985	$\frac{577}{408}$	0.0000021239
10	2378	$\frac{1393}{985}$	-0.0000003644
11	5741	$\frac{3363}{2378}$	0.0000000625
12	13860	$\frac{8119}{5741}$	-0.0000000107
13	33461	$\frac{19601}{13860}$	0.0000000018
14	80782	$\frac{47321}{33461}$	-0.0000000003
15	195025	$\frac{114243}{80782}$	5.4178×10^{-11}
16	470832	$\frac{275807}{195025}$	-9.2955×10^{-12}
17	1136689	$\frac{665857}{470832}$	1.5950×10^{-12}
18	2744210	$\frac{1607521}{1136689}$	-2.7364×10^{-13}
19	6625109	$\frac{3880899}{2744210}$	4.6948×10^{-14}