Exercise 2.1.

(a+b) Let $f\colon \mathbb{N}_0 \to \mathbb{Z}$ be given by $f(x) = x$, and $g\colon \mathbb{Z} \to \mathbb{N}_0$ be given by $g(x) := |x|$. Then $g \circ f$ is the identity map on $\mathbb{N}_0$: if $z \in \mathbb{N}_0$, then $(g \circ f)(z) = g(f(z)) = g(z) = |z| = z$. Hence $g \circ f$ is injective and surjective. But f is not surjective, and g is not injective. It follows that both assertions are wrong.

Exercise 2.2.

1. Let $\binom{\mathbb{N}}{n}$ denote the set of subsets of $\mathbb{N}$ of cardinality n. Then there is an injection $\iota_n\colon \binom{\mathbb{N}}{n} \to \mathbb{N}^n$ where ι_n is defined as follows: if S is a subset of $\mathbb{N}$ with n elements, then we order the elements of S in increasing order, i.e., $S = \{a_1, \ldots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$, and map S to $(a_1, \ldots, a_n)$. By Theorem 1.14(d) we know that $\mathbb{N}^n$ is enumerable via a function, say f. Then $g_n := \iota_n \circ f\colon \binom{\mathbb{N}}{n} \to \mathbb{N}$ is an enumerator for $\binom{\mathbb{N}}{n}$.

2. Let g_n be the enumerator for $\binom{\mathbb{N}}{n}$ constructed in the previous step. We now construct an injection $\varphi\colon A \to \mathbb{N} \times \mathbb{N}$ as follows: if S is a finite subset of $\mathbb{N}$ with t elements, then $\varphi(S) = (t, g_t(S))$. This mapping is injective: suppose that S and T are two different subsets of $\mathbb{N}$. If S and T have different numbers of elements, then $\varphi(S)$ and $\varphi(T)$ differ in their first coordinate, hence are different. If they have the same number, n, of elements, then $g_n(S)$ and $g_n(T)$ differ since g_n is an enumerator for $\binom{\mathbb{N}}{n}$, and hence $\varphi(S)$ and $\varphi(T)$ are different.

3. We construct directly an injection τ of A into $\mathbb{N}$: if $S = \{s_1, \ldots, s_t\}$ is a finite subset of $\mathbb{N}$, then we define $\tau(S) := \sum_{i=1}^{t} 2^{s_i}$. Because binary representations of natural numbers are unique, this mapping is an injection (in fact, even a bijection).

Exercise 2.3.

1. We have $(x \Leftrightarrow y) = (x \Rightarrow y) \wedge (y \Rightarrow x)$. Therefore

x	y	$x \Rightarrow y$	$y \Rightarrow x$	$x \Leftrightarrow y$
0	0	1	1	**1**
0	1	1	0	**0**
1	0	0	1	**0**
1	1	1	1	**1**

,

so that $(x \Leftrightarrow y) = \neg(x \oplus y)$.

2. We have

x	y	$x \Rightarrow y$	$\neg y \Rightarrow \neg x$	$(x \Rightarrow y) \Leftrightarrow (\neg y \Rightarrow \neg x)$
0	0	1	1	**1**
0	1	1	1	**1**
1	0	0	0	**1**
1	1	1	1	**1**

,

which proves the assertion.

3. From the truth table of $x \Rightarrow y$ we find that the polynomial representation of $x \Rightarrow y$ is $1 + x + xy$. Hence, the polynomial representation of $(x \Rightarrow (y \Rightarrow z))$ is that of $x \Rightarrow (1 + y + yz)$ which is $1 + x + x(1 + y + yz) = 1 + xy + xyz$. On the other hand, the polynomial representation of $((x \wedge y) \Rightarrow z)$ is $1 + xy + xyz$, which is the same as that of $(x \Rightarrow (y \Rightarrow z))$. Hence, the assertion follows.

4. This is not a tautology. Suppose that $x = z = 0, y = 1$. Then $(x \wedge y) \Rightarrow z$ is $0 \Rightarrow 0$ which is 1. Moreover, $(x \vee y) \Rightarrow z$ is $1 \Rightarrow 0$ which is 0. But $1 \Rightarrow 0$ is 0, and not 1. Hence, the formula is not a tautology.

Exercise 2.4. We use induction on n. If $n = 1$, then $f(x_1) = x_1$ and this is 0 iff $x_1 = 0$, i.e., iff an even number (namely 0) of x_i is 1. Let now $n \geq 2$. Let $g(x_1, \ldots, x_{n-1}) := x_1 \oplus \cdots \oplus x_{n-1}$, so that $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_{n-1}) \oplus x_n$. Using the truth table of $\oplus$ in Section 2.4.3, we see that $f(x_1, \ldots, x_n) = 0$ iff either $g(x_1, \ldots, x_{n-1}) = x_n = 0$, or $g(x_1, \ldots, x_{n-1}) = x_n = 1$. By induction hypothesis, $g(x_1, \ldots, x_{n-1}) = 0$ iff an even number of the x_i's is 1. Hence, in the former case, an even number of the x_i's among $x_1, \ldots, x_{n-1}$ is 1, and $x_n = 0$, so that an even number of the x_i's among $x_1, \ldots, x_n$ is 1. In the latter case, an odd number of the x_i's among $x_1, \ldots, x_{n-1}$ is 1, and $x_n = 1$, so that again an even number of the x_i's among $x_1, \ldots, x_n$ is 1.

Exercise 2.5.

(a) We have

$$\begin{aligned} f^{-1}(0) &= \{(0,0,0,0),(0,0,0,1),(0,0,1,0),(0,1,0,0),(0,1,1,1) \\ &= \quad (1,0,0,0),(1,0,1,1),(1,1,0,1),(1,1,1,0),(1,1,1,1)\}. \end{aligned}$$

Therefore, we obtain the following CNF for f:

$$\begin{aligned} f(x_1,x_2,x_3,x_4) &= (x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (x_1 \vee x_2 \vee x_3 \vee \neg x_4) \wedge (x_1 \vee x_2 \vee \neg x_3 \vee x_4) \wedge \\ & (x_1 \vee \neg x_2 \vee x_3 \vee x_4) \wedge (x_1 \vee \neg x_2 \vee \neg x_3 \vee \neg x_4) \wedge \\ & (\neg x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee \neg x_3 \vee \neg x_4) \wedge \\ & (\neg x_1 \vee \neg x_2 \vee x_3 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3 \vee x_4) \wedge \\ & (\neg x_1 \vee \neg x_2 \vee \neg x_3 \vee \neg x_4). \end{aligned}$$

(b) We have

$$\begin{aligned} f^{-1}(1) &= \{(1,1,0,0),(1,0,1,0),(1,0,0,1) \\ &= \quad (0,1,1,0),(0,1,0,1),(0,0,1,1)\}. \end{aligned}$$

Therefore, we obtain the following DNF for f:

$$\begin{aligned} f(x_1,x_2,x_3,x_4) &= (x_1 \wedge x_2 \wedge \neg x_3 \wedge \neg x_4) \vee (x_1 \wedge \neg x_2 \wedge x_3 \wedge \neg x_4) \vee \\ & (x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge x_4) \vee (\neg x_1 \wedge x_2 \wedge x_3 \wedge \neg x_4) \vee \\ & (\neg x_1 \wedge x_2 \wedge \neg x_3 \wedge x_4) \vee (\neg x_1 \wedge \neg x_2 \wedge x_3 \wedge x_4). \end{aligned}$$

(c) The answer is

$$\begin{aligned} f(x_1, x_2, x_3, x_4) &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 + \\ &\quad x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 + \\ &\quad x_1x_2x_3x_4, \end{aligned}$$

because of the following: if (x_1, x_2, x_3, x_4) has at most one nonzero entry, then all the terms vanish, and the value of the sum is zero. If (x_1, x_2, x_3, x_4) has exactly two coordinates equal to one, then exactly one term in the first group is nonzero, and all the other terms are zero, so the value of f on this tuple is one. If (x_1, x_2, x_3, x_4) has three nonzero coordinates, then the value of the sum in the first group is 1 (exactly three of the terms are nonzero), the value of the sum in the second group is also 1 (exactly one of the three terms is one), and the value of the last term is zero, so in total the value of f is $1+1=0$. If all the entries of (x_1, x_2, x_3, x_4) are 1, then all the terms are equal to 1, and since there is an even number of such terms, the value of f is 0. Hence, $f(x_1, x_2, x_3, x_4) = 1$ iff the vector (x_1, x_2, x_3, x_4) has exactly two nonzero components.