Exercise 4.1.

1. Reflexivity: we have $A \sim_C A$: $A = IAI^{-1}$ where I is the identity matrix.
2. Symmetry: suppose that $A \sim_C B$, say $TAT^{-1} = B$. Then $T^{-1}BT = A$, so $B \sim_C A$.
3. Transitivity: Suppose that $A \sim_C B$, say $TAT^{-1} = B$, and $B \sim_C D$, say $LBL^{-1} = D$. Then $LTAT^{-1}L^{-1} = (LT)A(LT)^{-1} = D$, so $A \sim_C D$.

Exercise 4.2. We use induction on the number of elements of the base set X of the relation R. If X has only one element, then R consists of one element only and the graph of R has only one edge which is a self-cycle, hence is a 1-clique. Suppose now that the assertion is true for all X with at most $m-1$ elements. Let X be a set with m elements. Let x be an element in X and consider all $y \in X$ such that $(x,y) \in R$. Call this set S. The graph (S,E) for which $E = \{(s,t) \in S \times S \mid (s,t) \in R\}$ is a $|S|$-clique with selfloops: all $(s,s) \in R$ for $s \in S$, hence the selfloops. Moreover, for all $s,t \in S$ we have that $(s,t) \in R$, since $(s,x) \in R$ and $(x,t) \in R$ by definition of S, and hence $(s,t) \in R$ by the transitivity of R. So the graph (S,E) is an $|S|$-clique.

By the definition of S we have $R \subseteq S \times S \cup (X-S) \times (X-S)$. Let $X' := X - S$, and $R' := R \cap X' \times X'$. Then R' is an equivalence relation on X': it is reflexive and symmetric, since the same is true for R. Moreover, if $x,y,z \in X'$ such that $(x,y) \in R'$ and $(y,z) \in R'$, then $(x,z) \in R'$ because the same is true for R. The graph of R is a disjoint union of (S,E) and the graph of R'. The latter is a disjoint union of cliques with selfloops, and so is (S,E). This proves the assertion.

Exercise 4.3.

(a) The relation is not reflexive if $A \in P(X)$: for example, if $A = \emptyset$, then $(\emptyset, \emptyset) \notin R$. However, if $A \in P(X) - \emptyset$, then $A \cap A = A \neq \emptyset$, so that $(A,A) \in R$.

(b) This is clearly the case since $A \cap B = B \cap A$, so that these sets are both empty or not at the same time.

(c) The relation is not transitive. Suppose that $B = A \sqcup C$, wherein $A, C \neq \emptyset$, and $A \cap C = \emptyset$. Then $(A,B) \in R$, and $(B,C) \in R$, but $(A,C) \notin R$.

Exercise 4.4. In both cases R is transitive by definition and we can show easily using the tansitivity that it is also reflexive. The problem is thus the symmetry. In the first case, we can show that all the odd number have to be in the same equivalence class and all the even number have to be in the same equivalence class. But this is not sufficient to be sure that it is an equivalent relation because all the odd number can be in relation with all the even number and not the contrary. So the symmetry is not imposed by the first condition.

In the second case, we can show that all numbers are in relation. The relation is then trivialy an equivalence relation.

Exercise 4.5. Let $A := \{a, b\}$ and $C := \{c, d\}$. In this case a relation $R \subseteq A \times C$ is not a function iff either it is not total, i.e., there is $x \in A$ such that for all $y \in C$ we have $(x, y) \notin R$, or there exists $x \in R$ such that $|\{y \in C \mid (x, y) \in R\}| > 1$. The following is a list of all relations in the first category:

$$\emptyset, \{(a, c)\}, \{(a, d)\}, \{(a, c), (a, d)\}, \{(b, c)\}, \{(b, d)\}, \{(b, c), (b, d)\}.$$

The following is a list of all relations in the second category:

$$\{(a, c), (a, d), (b, c)\}, \{(a, c), (a, d), (b, d)\}, \{(a, c), (b, c), (b, d)\},$$
$$\{(b, c), (b, d), (a, d)\}, \{(a, c), (a, d), (b, c), (b, d)\}.$$

Therefore, there are 11 relations that are not functions.

Exercise 4.6.

1. The set X is finite, and so is $P(X \times X)$. Since $R^n \in P(X \times X)$ for all m, it follows that in the sequence $R, R^2, R^3, \ldots$ not all the sets are distinct. Hence, there exist s, r, $s < r$, such that $R^s = R^r$.
2. Let $R = \{(1, 2), (2, 1)\} \subset \{1, 2\} \times \{1, 2\}$. Then $R^2 = R \circ R = \{(1, 1), (2, 2)\}$, and $R^3 = R \circ R^2 = R$. In general, we have $R^{2n} = R^2$ and $R^{2n-1} = R$ for $n \geq 1$.
3. Let $R = \{(a, a + 1) \mid a \in \mathbb{Z}\}$. Then $R^n = \{(a, a + n) \mid a \in \mathbb{Z}\}$, hence the R^n are distinct for all $n \in \mathbb{N}$.
4. Using Definition 3.3 we see that R is transitive iff $R^2 \subseteq R$. It follows that if R is transitive, then $R^3 = R \circ R^2 \subseteq R \circ R = R^2 \subseteq R$, and hence by induction $R^n \subseteq R$ for all $n \in \mathbb{N}$. Conversely, suppose that $R^n \subseteq R$ for all $n \in \mathbb{N}$. Then $R^2 \subseteq R$ as well, and by Definition 3.3 R is transitive.

Exercise 4.7. Suppose that $x, y, z \in X$.

1. Reflexivity: $(x, x) \in R_1$ and $(x, x) \in R_2$ since R_1 and R_2 are equivalence relations, hence $(x, x) \in R_1 \cap R_2$.
2. Symmetry: if $(x, y) \in R_1 \cap R_2$, then $(x, y) \in R_i$ for $i = 1, 2$, hence $(y, x) \in R_i$ (since R_i is symmetric), and hence $(y, x) \in R_1 \cap R_2$.
3. Transitivity: if $(x, y), (y, z) \in R_1 \cap R_2$, then $(x, y), (y, z) \in R_i$ for $i = 1, 2$, and hence $(x, z) \in R_i$, since R_i is transitive. Hence, $(x, z) \in R_1 \cap R_2$.

Suppose that $R_1 = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$ and $R_2 = \{(2, 2), (3, 3), (2, 3), (3, 2)\}$. Then R_1 and R_2 are equivalence relations. We have $(1, 2), (2, 3) \in R_1 \cup R_2$, but $(1, 3) \notin R_1 \cup R_2$.