Exercise 6.1.

1. Suppose that $\mathcal{P}$ does not have a maximal element. Let a_0 be an element of $\mathcal{P}$. Since it cannot be maximal, there is an element $a_1 > a_0$. Since a_1 is not maximal, there is an element $a_2 > a_1$, etc. The sequence $a_0 < a_1 < a_2 < \cdots$ contains two equal elements a_i and a_j, $j > i$, since $\mathcal{P}$ is finite. But this is a contradiction: by associativity, $a_i < a_{i+1} < \cdots < a_{j-1} = a_i$ implies that $a_i = a_{i+1} = \cdots = a_{j-1}$, a contradiction. Hence, $\mathcal{P}$ must have a maximal element. The assertion on the minimal element is proved analogously.

2. for $c, d \in X$ let $[c, d]$ denote the set $\{x \in X \mid c \leq x \leq d\}$. Consider the finite poset $\mathcal{P}_1 := \{x \in [a, b] \mid a < x\}$. By the previous part there exists a minimal element c_1 of $\mathcal{P}_1$. Then c_1 is a direct successor of a, since otherwise, if there is d such that $a < d < c_1$, then $a < d \leq b$, so $d \in \mathcal{P}_1$, contradicting the minimality of c_1. Moreover, $|\mathcal{P}_1| < |\mathcal{P}_0|$, where $\mathcal{P}_0 := [a, b]$, since $a \in \mathcal{P}_0 - \mathcal{P}_1$. Inductively, we define for $i \geq 2$ the poset $\mathcal{P}_i := \{x \in \mathcal{P}_{i-1} \mid c_{i-1} < x\}$, and c_i as a minimal element of $\mathcal{P}_i$. In the same way as above, we see that c_i is a direct successor of c_{i-1}. Since $c_{i-1} \in \mathcal{P}_{i-1} - \mathcal{P}_i$, the sizes $|\mathcal{P}_i|$ decrease, and hence there is some n such that $\mathcal{P}_{n+1} = \{b\}$, and $\mathcal{P}_{n+2} = \emptyset$. This proves that $a \to c_1 \to c_2 \to \cdots \to c_n \to b$, and proves the theorem.

Exercise 6.2. Let a and b be two minimal elements of the lattice $\mathcal{L}$. Let c be their infimum. Since a is minimal and $c \leq a$ by definition, we have that $c = a$. Similarly, $c = b$, so $a = b$. Therefore, minimal elements are unique. The assertion on maximal elements is proved analogously.

Exercise 6.3. Let N be the set of all n for which $P(n) = 0$, and suppose that $N \neq \emptyset$. Then N has a minimal element, call it m. Since $P(1) = 1$ by assumption, $m > 1$. By the definition of N and m, $P(m-1) = 1$. But the second hypothesis on P implies that $P(m) = 1$, so $m \notin N$, a contradiction. Hence N is empty.

Exercise 6.4. Let $n = \prod_{i=0}^{t-1} p_i$, where $p_0, \ldots, p_{t-1}$ are pairwise different primes. Then $\mathrm{Div}(n) = \{\prod_{i=0}^{t-1} p_i^{e_i} \mid e_i \in \{0, 1\}\}$. Define the mapping $f\colon \mathcal{B}_t \to \mathrm{Div}(n)$ by $f(S) = \prod_{s \in S} p_i^s$. This is a bijective map: if $d = \prod_{i=0}^{t-1} p_i^{e_i}$ and $S := \{i \mid e_i = 1\}$, then $f(S) = d$, so f is surjective. On the other hand, f is injective since the prime factorization of integers is unique (up to ordering of primes). Moreover, $S_1 \subseteq S_2$ iff $f(S_1) \mid f(S_2)$. Hence, f is a bijection between the posets $(\mathcal{B}_t, \subseteq)$ and $(\mathrm{Div}(n), |)$. The value $\mu(n)$ is the value of $\mu_{\mathcal{L}}(1, n)$ by definition, where $\mathcal{L} = (\mathrm{Div}(n), |)$. Because of the bijection f, this is the same as the value of $\mu_{\mathcal{L}'}(\emptyset, \underline{t})$, where $\mathcal{L}' = (\mathcal{B}_t, \subseteq)$. By Theorem 4.21.(2), the latter is equal to $(-1)^{|\underline{t}|} = (-1)^t$.