Exercise 7.1. The maximal antichain has already been discussed in the course and could consist of all subsets of size 2 of $\underline{5}$. To construct a minimal decomposition in chains, we need to make sure that each chain in this decomposition contains exactly one of the elements of the maximal antichain we have selected. Another maximal antichain of $P(\underline{5})$ is given by subsets of size 3, so every chain in the minimal decomposition must contain exactly one subset of size 3 and one subset of size 2. The difficulty in this problem lies in establishing a matching between subsets of size 2 and those of size 3, i.e., a bijective mapping between subsets of size 2 and those of size 3 of $\underline{5}$. In this particular case, this matching can be found by trial and error, though there is also a general algorithm for this task which we have not discussed in the course. Here is a solution in which each line corresponds to a different chain; the columns correspond to subsets of various sizes of $\underline{5}$.

$$\begin{array}{ccccccccccc}
\emptyset & \subseteq & \{0\} & \subseteq & \{0,1\} & \subseteq & \{0,1,2\} & \subseteq & \{0,1,2,3\} & \subseteq & \{0,1,2,3,4\} \\
 & & \{2\} & \subseteq & \{0,2\} & \subseteq & \{0,2,3\} & \subseteq & \{0,2,3,4\} & & \\
 & & \{3\} & \subseteq & \{0,3\} & \subseteq & \{0,3,4\} & \subseteq & \{0,1,3,4\} & & \\
 & & \{4\} & \subseteq & \{0,4\} & \subseteq & \{0,1,4\} & \subseteq & \{0,1,2,4\} & & \\
 & & \{1\} & \subseteq & \{1,2\} & \subseteq & \{1,2,4\} & \subseteq & \{1,2,3,4\} & & \\
 & & & & \{1,3\} & \subseteq & \{0,1,3\} & & & & \\
 & & & & \{1,4\} & \subseteq & \{1,3,4\} & & & & \\
 & & & & \{2,3\} & \subseteq & \{1,2,3\} & & & & \\
 & & & & \{2,4\} & \subseteq & \{1,2,4\} & & & & \\
 & & & & \{3,4\} & \subseteq & \{2,3,4\} & & & &
\end{array}$$

Exercise 7.2. The answer is $\lceil n/2 \rceil$ and it can be obtained as follows: first, note that the set $\{\lceil n/2 \rceil + 1, \ldots, n\}$ is an antichain, so that the width of $(\underline{n}, |)$ is at least this number. For every odd integer ℓ in $\{3, \ldots, n\}$, the chain C_ℓ is defined as $C_\ell = \{2^j \ell \mid 0 \le j \le \lfloor \log_2(n/\ell) \rfloor\}$. Additionally, we define $C_1 := \{2^j \mid 0 \le j \le \lfloor \log_2(n) \rfloor\} \cup \{0\}$. These chains are disjoint by the unique factorization theorem of the integers. Moreover, they cover all the elements of $\underline{n}$, since every nonzero such element can be written as $\ell 2^j$ for an odd number ℓ and a non-negative integer j. The number of these chains is the number of odd numbers in $\{1, \ldots, n\}$ which is $\lceil n/2 \rceil$. It follows by the theorem of Dilworth that the width of $(\underline{n}, |)$ is at most $\lceil n/2 \rceil$ and proves the result.

Exercise 7.3. Suppose that all chains in $\mathcal{P}$ are of length $\le a$. Let w be the width of $\mathcal{P}$. By the theorem of Dilworth, there are w disjoint chains that cover $\mathcal{P}$. Since all these chains have length $\le a$, it follows that $wa \ge ab + 1$, so that $w \ge b + 1/a$, i.e., $w \ge b + 1$.

Exercise 7.4. Define the partial order on $\mathbb{R}^2$ by $(x, y) \le (x', y')$ iff $x \le x'$ and $y \le y'$. Let $\mathcal{P}$ be the poset $(X, \le)$, wherein $X = \{(x_1, y_1), \ldots, (x_n, y_n)\}$. Then polygonal paths with negative slope correspond to antichains in X, while polygonal paths with positive slope correspond to chains: (x, y) and (x', y') are incomparable iff they are not equal and either $x \le x'$ and $y \ge y'$ or $x \ge x'$ and $y \le y'$; in both cases the slope $(y - y')/(x - x')$ is negative, and conversely this number is negative iff (x, y) and (x', y') are incomparable. Let w be the width of $\mathcal{P}$. We proceed now the same way as in the previous exercise. Suppose that all chains in $\mathcal{P}$ are of length $\le a$. Since there is a decomposition of $\mathcal{P}$ with w disjoint chains, we see that

$wa \geq ab + 1$, so $w \geq b + 1$.

Exercise 7.5. Let $\{A_1, \ldots, A_k\}$ be a minial decomposition of $\mathcal{P}$ and let C be a maximal chain with p elements. If $k < p$, then there exist $c, c' \in C$, $c \neq c'$, such that c and c' belong to the same antichain. Since this is impossible, we have $k \geq p$.

To show that $k \leq p$, we construct a decomposition into p antichains; if there exists such a decomposition, then the minimal decomposition contains at most p antichains, and hence $k \leq p$. For $c \in \mathcal{P}$, let $\ell(c)$ be the size of a maximal chain in which the largest element is c. If $\ell(c) = \ell(c')$, then c and c' are incomparable: if they are comparable, say $c < c'$, then $\ell(c) < \ell(c')$. Hence, the sets

$$A_i := \{c \mid \ell(c) = i\}, \ \ 1 \leq i \leq p$$

form p antichains which cover $\mathcal{P}$.