Lecture 20
Properties of Belief Propagation II

20.1. Introduction

In this lecture, we want to show that under iterative belief-propagation, the densities of messages are symmetric given that the initial channel distribution is symmetric. In the second part, we will derive the stability condition for BEC and general BIMS channels.

20.2. Symmetric densities under density evolution

Theorem 20.1. *Given that the channel $\mathcal{C}$ is a BIMSC, then all densities in density evolution are symmetric, where the density evolution equation is given by*

$$f_{i+1} := f_{\mathcal{C}} \otimes \lambda(\Gamma^{-}1(\rho(\Gamma(f_i)))). \tag{20.1}$$

In this relation, $f_{\mathcal{C}}$ is the initial channel distribution coming from the BIMS channel $\mathcal{C}$, and Γ, Γ^{-1} are defined as in lecture 19. λ and ρ are the degree distributions on the variable and check node side respectively from an edge perspective.

We will proceed by induction in the following steps:

1. First, we argue that the output channel density $f_{\mathcal{C}}$ of a BIMS channel is symmetric (Thm. 20.2)
2. Then, if f_i is symmetric, one can show using Thm. 20.3, that symmetry is preserved under $\Gamma^{-1}(\rho(\Gamma(f_i)))$.
3. Furthermore, if $f_{\mathcal{C}}$ symmetric then f_{i+1} symmetric (Thm. 20.4).
4. Finally, we can conclude, that if the channel density $f_{\mathcal{C}}$ is symmetric then all densities in density evolution are symmetric.

Theorem 20.2. *If $\mathcal{C}$ is BIMSC with $f_{\mathcal{C}} = P(y|1)$ then $f_{\mathcal{C}}$ is symmetric*

Proof.

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- A channel is said to be symmetric if
$$P(y|-1) = P(-y|1). \tag{20.2}$$
- The loglikelihood ratio (LLR) is defined as
$$z = \ln\frac{P(y|1)}{P(y|-1)} \text{ (There exist exactly one value } y \text{ which satisfies this relation).} \tag{20.3}$$
and
$$e^z = \frac{P(y|1)}{P(y|-1)} \Leftrightarrow P(y|-1)e^z = P(y|1) \tag{20.4}$$

From Eq. 20.2 and 20.3, we can see that
$$\begin{aligned} z(y) &= \ln\frac{P(y|1)}{P(y|-1)} \\ &= \ln\frac{P(-y|-1)}{P(-y|1)} \\ &= \ln\left(\frac{P(-y|1)}{P(-y|-1)}\right)^{-1} \\ &= -z(-y) \end{aligned}$$

Finally
$$e^z f_{\mathcal{C}}(-z) = e^z P(y|1) = P(-y|1) = f_{\mathcal{C}}(z)$$
which concludes the proof

□

Theorem 20.3. *f: density on $\mathbb{R}$, then if f is symmetric $\Leftrightarrow \Gamma(f)$ is symmetric.*

Proof. We have seen in lecture 19, that
$$\Gamma(f)(x) = \chi_{s=1}\frac{f(-\ln\tanh(\frac{x}{2}))}{\sinh(x)} + \chi_{s=-1}\frac{f(\ln\tanh(\frac{x}{2}))}{\sinh(x)}$$
Using the symmetry condition for f, we can deduce that $\tanh(\frac{x}{2})f(-\ln\tanh(\frac{x}{2})) = f(\ln\tanh(\frac{x}{2}))$. As the sum of symmetric densities is again a symmetric density, $\Gamma(f)$ is symmetric. □

In the next step, we want to show that the convolution operation in eq.20.1 preserves symmetry. This is done in two steps: we first show the symmetry for convolution of densities over $\mathbb{R}$. In a second step, we show that it still holds for ($\mathbb{F}_2 \times \mathbb{R}_{\geq 0}$)

Theorem 20.4.

(a) *Given f_1, f_2 are symmetric densities on $\mathbb{R}$ then $f_1 \otimes f_2$ is also symmetric*

(b) *Given g, h symmetric densities on $\mathbb{F}_2 \times \mathbb{R}_{\geq 0}$ then so is $g \otimes h$.*

Proof.

(a)

$$\begin{aligned}(f_1 \otimes f_2)(-x) &= \int_{\mathbb{R}} f_1(y) f_2(-x-y)\mathrm{dy} \\ &= \int_{\mathbb{R}} f_1(-y)e^{y} f_2(x+y)e^{-x}e^{-y}\mathrm{dy} \\ &= e^{-x}\int_{\mathbb{R}} f_1(-y)f_2(x+y)\mathrm{dy} \\ &= e^{-x}(f_1 \otimes f_2)(x)\end{aligned}$$

(b) previously, we have defined g to be $g = \chi_{s=1}g_1 + \chi_{s=-1}g_{-1}$. Similarly $h = \chi_{s=1}h_1 + \chi_{s=-1}h_{-1}$. The convolution of g and h is the density u given by

$$\begin{aligned}\Rightarrow (g \otimes h) &= \chi_{s=1}(g_1 \otimes h_1 + g_{-1} \otimes h_{-1}) + \chi_{s=-1}(g_1 \otimes h_{-1} + g_{-1} \otimes h_1) \\ &= \chi_{s=1}u_1 + \chi_{s=-1}u_{-1}\end{aligned}$$

For showing that the densities remain symmetric under this convolution, we need to show that u satisfies $\tanh(\frac{x}{2})u_1(x) = u_{-1}(x)$. This can be done using the Hadamard-Walsh-Transform.

$$\begin{aligned}g_1(x) + g_{-1}(x) = g_1(x) + \tanh(\frac{x}{2})g_1(x) = &\quad g_1(x)(1 + \frac{e^x-1}{e^x+1}) = g_1(x)(\frac{e^x+1+e^x-1}{e^x+1}) = 2g_1(x)\frac{e^x}{e^x+1} \\ g_1(x) - g_{-1}(x) = g_1(x) - \tanh(\frac{x}{2})g_1(x) = &\quad g_1(x)(1 - \frac{e^x-1}{e^x+1}) = g_1(x)(\frac{e^x+1-e^x+1}{e^x+1}) = 2g_1(x)\frac{1}{e^x+1}\end{aligned}$$

$$\begin{aligned}t_1(y) = \frac{1}{2}((g_1+g_{-1}) \otimes (h_1+h_{-1}))(y) =& \frac{1}{2}((g_1\frac{e^x}{e^x+1}) \otimes (h_1\frac{e^x}{e^x+1}))(y) \\ =& \frac{4}{2}\int_{\mathbb{R}_{\geq \nvdash}} g_1(x)h_1(y-x)\frac{e^x e^{y-x}}{(e^x+1)(e^{y-x}+1)}dx \\ =& 2e^y\int_{\mathbb{R}_{\geq 0}} \underbrace{g_1(x)h_1(y-x)\frac{dx}{(e^x+1)(e^{y-x}+1)}}_{(a)}\end{aligned} \tag{20.5}$$

Similarly we get

$$t_2(y) = \frac{1}{2}(g_1 - g_{-1}) \otimes (h_1 - h_{-1}) = 2\int(a) \tag{20.6}$$

By the distribution law of convolution

$$\begin{aligned}u_1(y) &= t_1(y) + t_2(y) \\ u_{-1}(y) &= t_1(y) - t_2(y) \\ u_1(y) &= 2(e^y+1)\int(a) \\ u_{-1}(y) &= 2(e^y-1)\int(a)\end{aligned}$$

and finally

$$u_1(y)\tanh(\frac{y}{2}) = u_1(y)\frac{e^y-1}{e^y+1} = \frac{e^y-1}{e^y+1}2(e^y+1)\int(a) = u_{-1}(y)$$

□

We know, that for general densities, the operation $\rho(f)$ is given by summing over all check node degrees: $\sum \rho(f)^{\otimes(i-1)}$.

As symmetry is preserved under convolution, $\rho(\lambda(f_i))$ is symmetric. Part b of Thm. 20.4 implies then that $\lambda^{-1}(\rho(\lambda(f_i)))$ is symmetric. And finally, we can conclude that Eq. 20.1 is symmetric.

20.3. Fixed points and stability conditions of Density Evolution

Let $f(x)$ be a symmetric distribution. Then we define the error probability as

$$P_e(f) = \int_{-\infty}^{0} f(x)\mathrm{dx}$$

From this, the following corollary can be noted.

Corollary 20.5. *If $f(x)$ is a symmetric distribution then $P_e(f) \to 0$ if and only if $f \to \Delta_\infty$*

For deriving the stability conditions, we will look again at the density evolution equation:

$$f_{i+1} = f_{\mathcal{C}} \otimes \lambda(\Gamma^{-1}(\rho(\Gamma(f_i))))$$

20.3.1. Fixed points and stability condition for BEC channel

First for the BEC case:

Definition 20.6 (Description of threshold in terms of fixed points)**.** Denote the remaining number of erasures by P_i and let P_{i+1} be the remaining number of erasures after a further iteration. The threshold is given by the maximum number P_0 such that the equation $P_{i+1} = P_i$ has no solution for $P_i \in (0, P_0)$.

An alternative description can be given by the idea that for successful decoding, we want $P_{i+1} < P_i \; \forall \; 0 < P_i < P_0$.

Obviously, this expression can be written as

$$\begin{aligned} P_{i+1} = p\,\lambda(1-\rho(1-P_i)) &\leq P_i \\ \Leftrightarrow p\,\lambda(1-\rho(1-x)) - x &\leq 0 \end{aligned}$$

where we replaced P_i by x and P_0 by p. Furthermore, lets call $p\,\lambda(1-\rho(1-x)) - x = h(x)$.

The *stability condition* of the fixed point $x = 0$ is now be given by

$$\frac{d}{dx} h|_{x=0} \leq 0$$

$$h'(0) = p\,\rho'(1)\lambda'(0) - 1 \leq 0$$

Then $p_e \to 0$ if $\rho'(1)\lambda'(0) \leq \frac{1}{p}$ is fulfilled.

20.4. Stability condition of general BIMS channels

We consider now transmission over a general BIMS channel. One can observe that if $P_i = \Delta_\infty$ then also $P_{i+1} = \Delta_\infty$, i.e. Δ_∞ is a fixed point of density evolution. As we want error probability to go to zero for an infinite number of iterations, we want distribution to converge to Δ_∞. Thus, we want to proof that ones the error probability approaches zero, we will actually reach $P_e = 0$ without hitting another fix point.

Similarly to the BEC case, we will linearize the density evolution equation in order to analyze the local convergence to the fixed Δ_∞.

Consider a density

$$f = 2\epsilon \cdot g + (1 - 2\epsilon)\Delta_\infty,$$

where g is any symmetric density, but not Δ_∞. Furthermore, $P_e(f) = \epsilon$. We now replace g by the erasure probability, which increases the probability of error P_e.

$$f = 2\epsilon \cdot \Delta_0 + (1 - 2\epsilon)\Delta_\infty,$$

Based on the density evolution equation for general channels

$$f_{i+1} = f_{\mathcal{C}} \otimes \lambda(\Gamma^{-1}(\rho(\Gamma(f_i)))) = T(f_i),$$

we are now interested in $T(f)$, the density after one iteration.

$$\begin{aligned}
\Gamma(f) &= 2\epsilon\Delta_\infty + (1 - 2\epsilon)\Delta_0 \\
\rho(\Gamma(f)) &= \rho(1 - 2\epsilon)\Delta_0 + (1 - \rho(1 - 2\epsilon))\Delta_\infty \\
\Gamma^{-1}(\rho(\Gamma(f))) &= \rho(1 - 2\epsilon)\Delta_\infty + (1 - \rho(1 - 2\epsilon))\Delta_0 \\
\lambda(\Gamma^{-1}(\rho(\Gamma(f)))) &= \lambda(1 - \rho(1 - 2\epsilon))\Delta_0 + (1 - \lambda(1 - \rho(1 - 2\epsilon)))\Delta_\infty \\
T(f) &= \lambda(1 - \rho(1 - 2\epsilon))\Delta_0 + (1 - \lambda(1 - \rho(1 - 2\epsilon)))\Delta_\infty \\
T(f) &= \lambda(1 - \rho(1 - 2\epsilon))f_{\mathcal{C}} + (1 - \lambda(1 - \rho(1 - 2\epsilon)))\Delta_\infty, \qquad (20.7)
\end{aligned}$$

where in the last term, we replaced the Δ_0 again by the channel density $f_{\mathcal{C}}$. The right term of Eq. 20.7 can now again be expanded in Taylor series and this we obtain as a linear term:

$$T(f) = 2\epsilon(\lambda'(0)\rho'(1))f_{\mathcal{C}} + (1 - 2\epsilon\lambda'(0)\rho'(1))\Delta_\infty + O(\epsilon^2) \qquad (20.8)$$

This can be continued for 2 iterations

$$T(T(f)) = 2\epsilon(\lambda'(0)\rho'(1))^2 f_{\mathcal{C}}^{\otimes 2} + (1 - 2\epsilon(\lambda'(0)\rho'(1))^2)\Delta_\infty + O(\epsilon^2)$$

and generalized for n iterations

$$T^n(f) = 2\epsilon(\lambda'(0)\rho'(1))^n f_{\mathcal{C}}^{\otimes n} + (1 - 2\epsilon(\lambda'(0)\rho'(1))^n)\Delta_\infty + O(\epsilon^2) \qquad (20.9)$$

From Eq. 20.9, we expect that Δ_∞ is a fixed point and the probability of error going to zero, if the probability of error of $f_{\mathcal{C}}^{\otimes n}$ goes to zero. Thus, we want

$$(\lambda'(0)\rho'(1))^n f_{\mathcal{C}}^{\otimes n} \to 0. \qquad (20.10)$$

Lets describe the probability of error as

$$\begin{aligned}
\ln(P_e(f_{\mathcal{C}}^{\otimes n})) &= nc + O(n) \\
P_e(f_{\mathcal{C}}^{\otimes n}) &= e^{cn} \cdot e^{O(n)}
\end{aligned}$$

Then we can write Eq. 20.10 as following:

$$\lambda'(0)\rho'(1) \cdot e^c \leq 1$$

which is a stability condition similar to the one for the BEC channel and we can observe that stability is condition on the order of growth of e^c. It can be obtained by the following statements:

1. The limit $c := -\lim_{n\to\infty} \frac{\ln(P_e(f_{\mathcal{C}}^{\otimes n}))}{n}$ exists if $\int_{\mathbb{R}} f_{\mathcal{C}}(x)e^{-\frac{x}{2}}\mathrm{dx} < \infty$ for all s in the neighborhood of zero. If then, we assume that $\lambda'(0)\rho'(1)) < e^c$ then there exists an integer n such that $\lambda'(0)\rho'(1)) < P_e(f_{\mathcal{C}}^{\otimes n}) < 1$.

2. In a last step, we show that $e^{-c} := \int_{\mathbb{R}} f_{\mathcal{C}}(x)e^{-\frac{x}{2}}dx$

Proof. Using a general large deviation principle and the generating function of $f_{\mathcal{C}}$, the following relation holds:

$$e^{-c} = \inf_{s \leq o} \int_{\mathbb{R}} f_{\mathcal{C}}(x) e^{sx} \mathrm{dx}.$$

This can be rewritten as

$$= \inf_{s \leq o} \frac{1}{2} \left(\int_{\mathbb{R}} f_{\mathcal{C}}(x) e^{sx} + \int_{\mathbb{R}} f_{\mathcal{C}}(-x) e^{-sx} \mathrm{dx} \right)$$

$$` = \inf_{s \leq o} \frac{1}{2} \left(\int_{\mathbb{R}} f_{\mathcal{C}}(x) e^{-\frac{x}{2}} (e^{s(x+\frac{1}{2})} + e^{-s(x+\frac{1}{2})}) \mathrm{dx} \right)$$

and results finally in the Batthacharyya parameter

$$e^{-c} = \int_{\mathbb{R}} f_{\mathcal{C}}(x) e^{-\frac{x}{2}} dx.$$

□

20.5. Examples

We now give stability conditions for some common BIMS channels:

BEC $\rho'(1)\lambda'(0) < \frac{1}{p}$

BSC $\rho'(1)\lambda'(0) < \frac{1}{2\sqrt{p(1-p)}}$

AWGN(σ^2) $\rho'(1)\lambda'(0) < e^{\frac{1}{2\sigma^2}}$