Exercise Sheet 3

Exercise 3.1. Let $x \in \mathbb{F}_2^n$ be of weight d. What is the number of binary vectors of weight w that are orthogonal to x? (*Hint:* Use MacWilliams identities.)

Exercise 3.2. In this exercise, we will look at two common procedures for creating new codes from old ones.

1. **Puncturing.** Let C be an $[n,k,d]_q$-code. The *punctured code at position i*, denoted C^i, is the set of words $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$ of length $n-1$ formed by removing the ith coordinate of each codeword. Show that C^i is a linear code. What are its parameters?
2. **Shortening.** Let C be an $[n,k,d]_q$-code. The *shortened code* C_i is formed as follows: let C' be the intersection of C with the hyperplane $\{x \in \mathbb{F}_q^n : x_i = 0\}$. C_i is then formed by puncturing C' at position i. Show that C_i is a linear code. What are its parameters?
3. Show that
$$(C^{\perp})_i = (C^i)^{\perp}.$$

Exercise 3.3. The *extended Hamming code* is constructed as follows: start with the $[7,4,3]_2$-Hamming code and add a position to each codeword. In that position, put a 1 if the codeword is of odd weight, and put a 0 otherwise.

1. Show that the extended Hamming code is an $[8,4,4]_2$-code and find a generator and a check matrix for this code.
2. Show that the dual of the extended Hamming code is equal to the code itself.

Exercise 3.4. An $[n,k,d]_q$-code is called *perfect* if the Hamming balls of radius $(d-1)/2$ around the codewords form a disjoint union of $\mathbb{F}_q^n$. Show that binary Hamming codes are perfect.

Exercise 3.5. Given integers $N \le n$, $K \ge k$, and $D \ge d$, show that if there is no $[n,k,d]_q$-code, then there is no $[N,K,D]_q$-code.