Exercise Sheet 1

Exercise 1.1. Let n and k be integers such that $k \leq n$ and let G_1 be a $k \times r$ matrix, where $r = n - k$. Show that if $(I_k|G_1)$ is a generator matrix for a linear code C, then $(-G_1^\top|I_r)$ is a check matrix for C.

Exercise 1.2. Let n and k be integers such that $k \leq n$ and suppose that G and H, of dimensions $k \times n$ and $(n-k) \times n$, respectively, are generator and parity-check matrices for a linear code $C \subseteq \mathbb{F}_2^n$. Is the $n \times n$ matrix

$$\begin{pmatrix} G \\ H \end{pmatrix}$$

necessarily invertible? Prove or exhibit a counterexample.

Exercise 1.3. Show that any k-dimensional linear code $C \subseteq \mathbb{F}_2^n$ has

$$\prod_{i=1}^{k}(2^k - 2^{i-1}) = 2^{\binom{k}{2}} \prod_{i=1}^{k}(2^i - 1)$$

distinct generator matrices.

Exercise 1.4. Show that if C is a binary code of dimension k, then the codewords of even weight in C either form the whole code C or form a linear subcode of dimension $k-1$.

Exercise 1.5. The *International Standard Book Number* (ISBN) is a numeric book identifier used for the ease of handling books particularly by booksellers and libraries. According to the 2001 standard, a unique 9-digit identifier is assigned to each book (based on the language of the publishing country, publisher, and the title) and a check digit is then affixed to the identifier. The aim of the checksum is to facilitate detection of two common typing errors made in book handling: Typing a wrong digit and interchanging two subsequent digits. Taking the check digit into account, a valid ISBN can be regarded as a vector

$$x = (x_1, \ldots, x_{10})$$

where $x_2, \ldots, x_{10} \in \{0, \ldots, 9\}$ and $x_1 \in \{0, \ldots, 10\}$ is the checksum (we suppose that 10 can be represented by a special symbol X), computed according to the rule

$$\sum_{i=1}^{10} i x_i = 0 \mod 11.$$

1. Show that the ISBN code can detect a single error.
2. Show that it can detect transposition of any digit with an adjacent digit.
3. What is the minimum distance of this code?
4. If we used the simpler rule

   $$\sum_{i=1}^{10} x_i = 0 \mod 11$$

   instead of the one above, could the code still detect errors? What about transpositions?