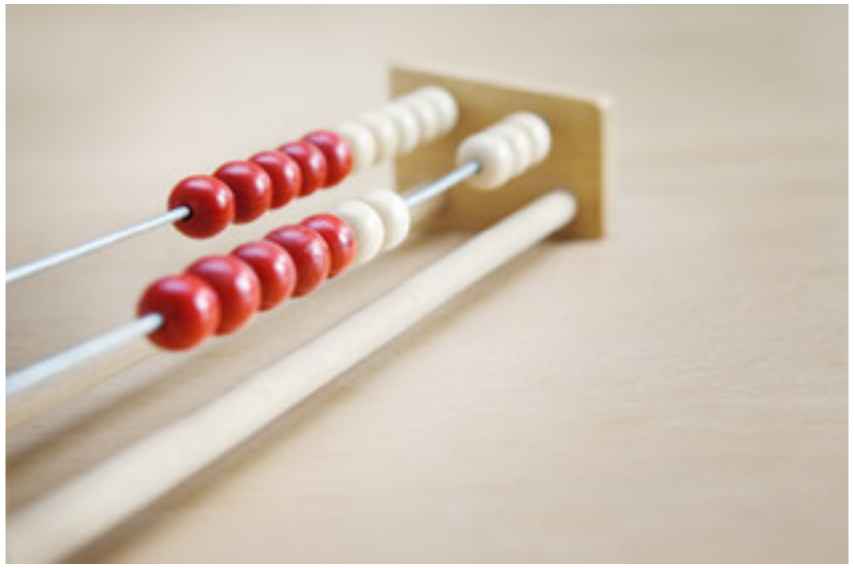
Lecture 18
Counting



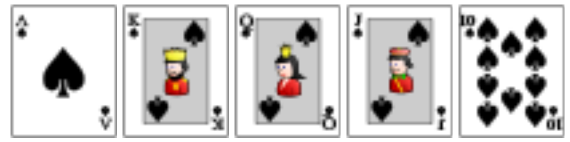
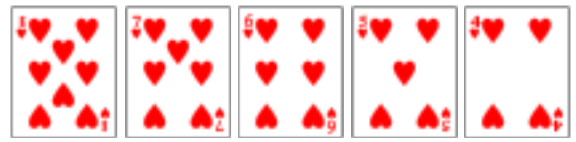
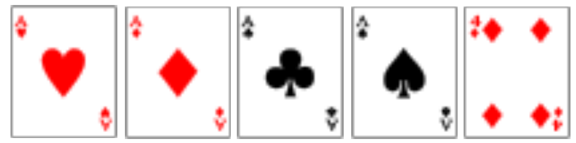
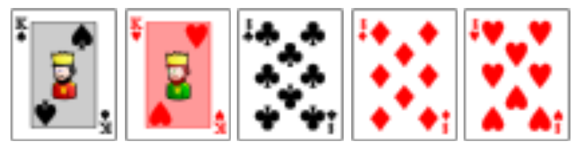
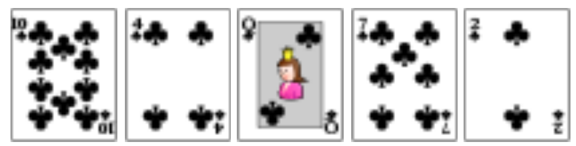
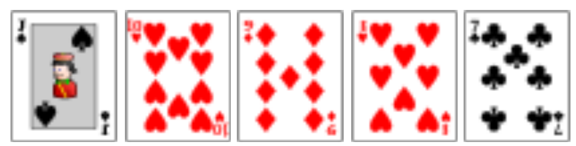
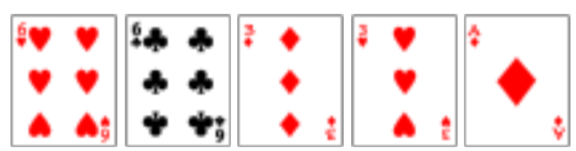
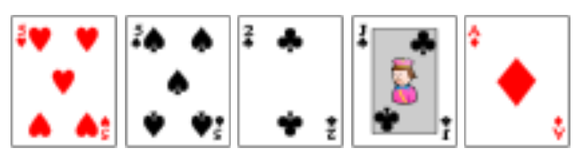
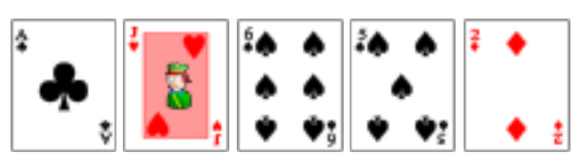
algolma
laboratoire d'algorithmique
laboratoire de mathématiques algorithmique

Poker Hands

Game of (Straight) Poker

- Each of the players is dealt five cards from a deck of 52 cards
- Players can switch some of their cards against new ones
  - We are not going to talk about this here.
- Players bet on their final hands
- Cards are revealed
- Highest hand wins

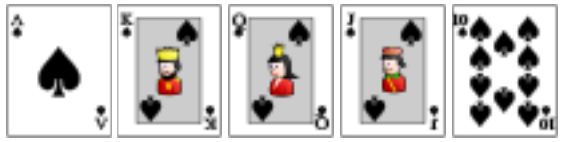
Winning Hands

Royal flush		5 consecutive cards of one color, starting with 10
Straight flush		5 consecutive cards of one color, starting with ≤ 9
Four of a kind		Four cards are of one kind
Full house		A pair of one kind, three cards of another kind
Flush		All cards of one color, but not consecutive
Straight		Consecutive cards, not all of the same suit
Three of a kind		Three colors of the same kind, the other two different
Two pairs		Two pairs of same kind, but not a four
One pair		One pair of the same kind, the other three different
No pair		None of the above

Total number of distinct hands

$$C(52,5) = \frac{52!}{5!47!} = \mathbf{\underline{2'598'960}}$$

Number of royal flush hands



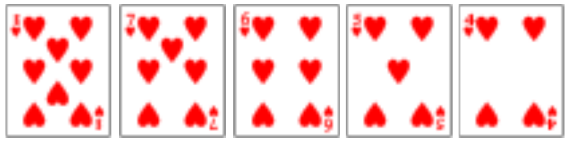
5 consecutive cards of one suit, starting with 10

ways to color the hand

$$1 * 4 = \underline{4}$$

ways to pick the five cards without regard for the suit

Number of straight flush but not royal flush hands



5 consecutive cards of one color, starting with ≤ 9

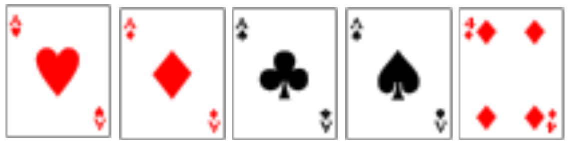
ways to color the hand

$$9 * 4 = \underline{\mathbf{36}}$$

The first card in the royal flush hand can be A, 2, 3, 4, 5, 6, 7, 8, 9, so 9 possibilities

Cumulative count
4 + 36 40

Number of four of a kind hands



Four cards are of one kind

\# ways to pick the kind of the last card. Cannot be equal to kind of first

$$C(13,1) * C(12,1)*4 = \frac{13!}{1!12!} \cdot \frac{12!}{1!11!} \cdot 4$$

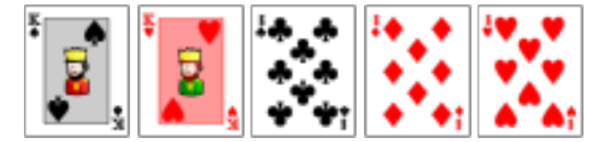
\# ways to pick one of 13 kinds

\# ways to color last card

= 624

Cumulative count
4
+ 36
+ 624
664

Number of full house hands



A pair of one kind, three cards of another kind

ways to color the three cards.

ways to color the last two cards

$$C(13,1) * C(4,3) * C(12,1) * C(4,2) = \frac{13!}{1!12!} \cdot \frac{4!}{3!1!} \cdot \frac{12!}{1!11!} \cdot \frac{4!}{2!2!}$$

ways to pick one of 13 kinds

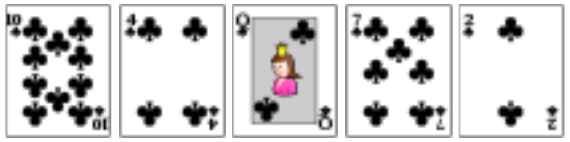
ways to pick a kind out of the remaining 12 for the last two cards

= 3’744

Cumulative count

4
+ 36
+ 624
+ 3’744
4’408

Number of flush but not royal/straight flush hands



All cards of one color, but not consecutive

ways to color the hand

$$C(13,5) * C(4,1) - C(10,1)*4 = \frac{13!}{5!8!} \cdot \frac{4!}{1!3!} - 40$$

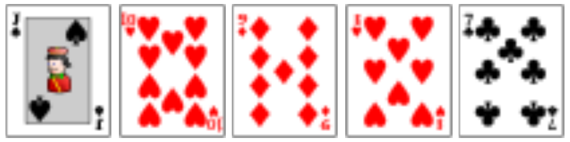
ways to pick the five kinds making up the hand

royal/straight flush hands

= <u>5’108</u>

Cumulative count
4
+ 36
+ 624
+ 3’744
+ 5’108
9’516

Number of straight but not flush/royal flush hands



Consecutive cards, not all of the same suit

ways to color the hand

$$10 * C(4,1)^5 - C(10,1)*4 = 10 \cdot \left(\frac{4!}{1!3!}\right)^5 - 40$$

ways to pick the five kinds making up the hand

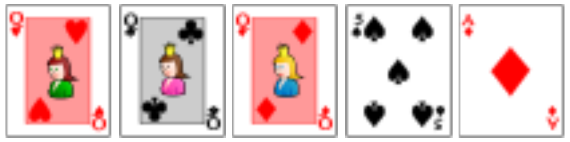
royal/straight flush hands

= <u>10’200</u>

Cumulative count

4
+ 36
+ 624
+ 3’744
+ 5’108
+ 10’200
19’716

Number of hands with three of a kind



Three colors of the same kind, the other two different

ways to color the three cards.

ways to color the last 2 cards

$$C(13,1) * C(4,3) * C(12,2)*4^2 = \frac{13!}{1!12!} \cdot \frac{4!}{3!1!} \cdot \frac{12!}{2!10!} \cdot 4^2$$

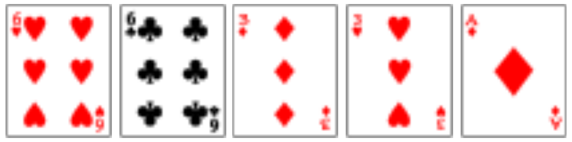
ways to pick 1 of 13 kinds to make the “three of a kind”

ways to pick the other two cards. They cannot be a pair, and have to be of a different kind.

= 54’912

Cumulative count
4
+ 36
+ 624
+ 3’744
+ 5’108
+ 10’200
+ 54’912
74’628

Number of hands with two distinct pairs



Two pairs of same kind, but not a four

ways to color the two pairs. Each pair has independently C(4,2) ways

ways to color the last card

$$C(13,2) * C(4,2)^2 * C(11,1)*4 = \frac{13!}{2!11!} \cdot \left(\frac{4!}{2!2!}\right)^2 \cdot \frac{11!}{1!10!} \cdot 4$$

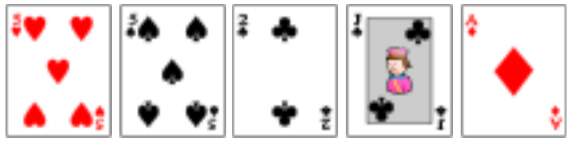
= 123’552

ways to pick 2 of 13 kinds to make the two pairs

ways to pick a kind out of the remaining 11 kinds for the last card

Cumulative count
4
+ 36
+ 624
+ 3’744
+ 5’108
+ 10’200
+ 54’912
+ 123’552
198’180

Number of hands with exactly one pair



One pair of the same kind, the other three different

ways to pick two distinct colors for the two cards in the pair

ways to pick the colors of the remaining three cards

$$C(13,1) * C(4,2) * C(12,3)*4^3 = \frac{13!}{1!12!} \cdot \frac{4!}{2!2!} \cdot \frac{12!}{3!9!} \cdot 4^3$$

= 1’098’240

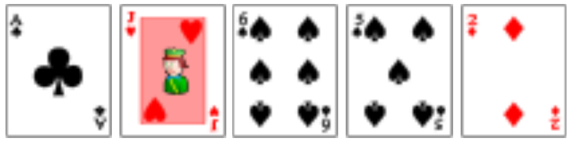
ways to pick one of 13 kinds to make a pair

ways to pick 3 distinct kinds out of the remaining 12 for the other three cards in the hand

Cumulative count

4
+ 36
+ 624
+ 3’744
+ 5’108
+ 10’200
+ 54’912
+ 123’552
+ 1’098’240
1’296’420

None of the above



None of the above

\# hands counted so far

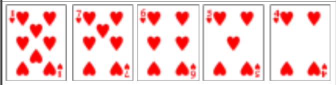
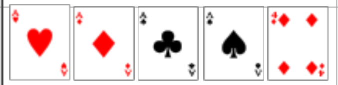
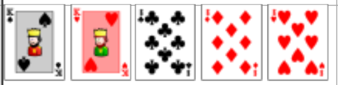
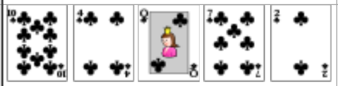
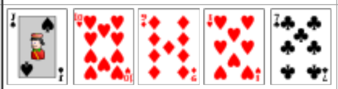
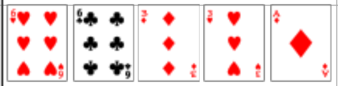
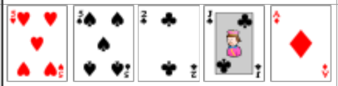
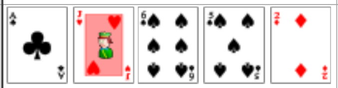
$$C(52,5) - 1'296'420 = \underline{\mathbf{1'302'540}}$$

\# all hands

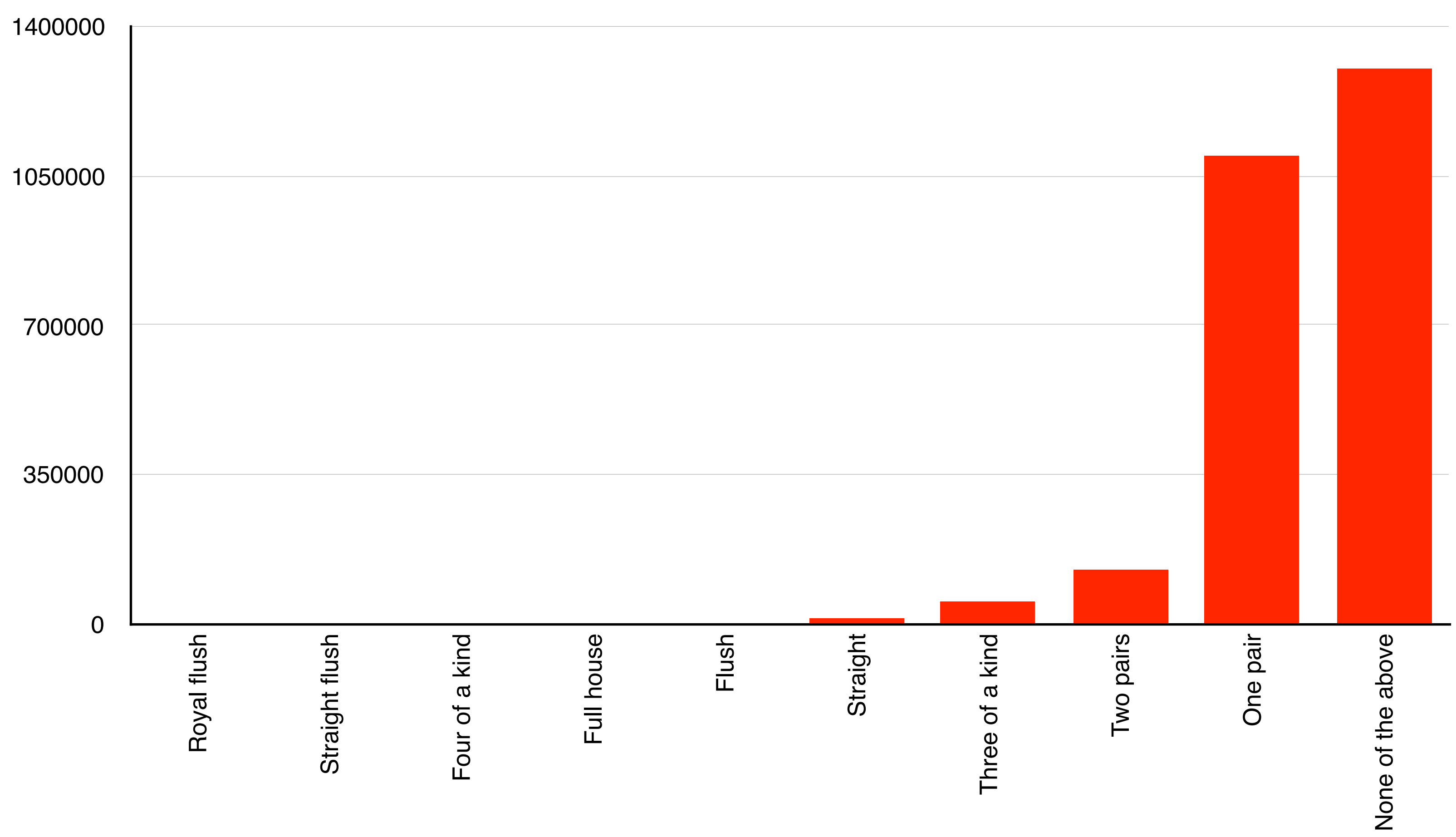
Cumulative count

4
+ 36
+ 624
+ 3’744
+ 5’108
+ 10’200
+ 54’912
+ 123’552
+ 1’098’240
1’296’420

Results

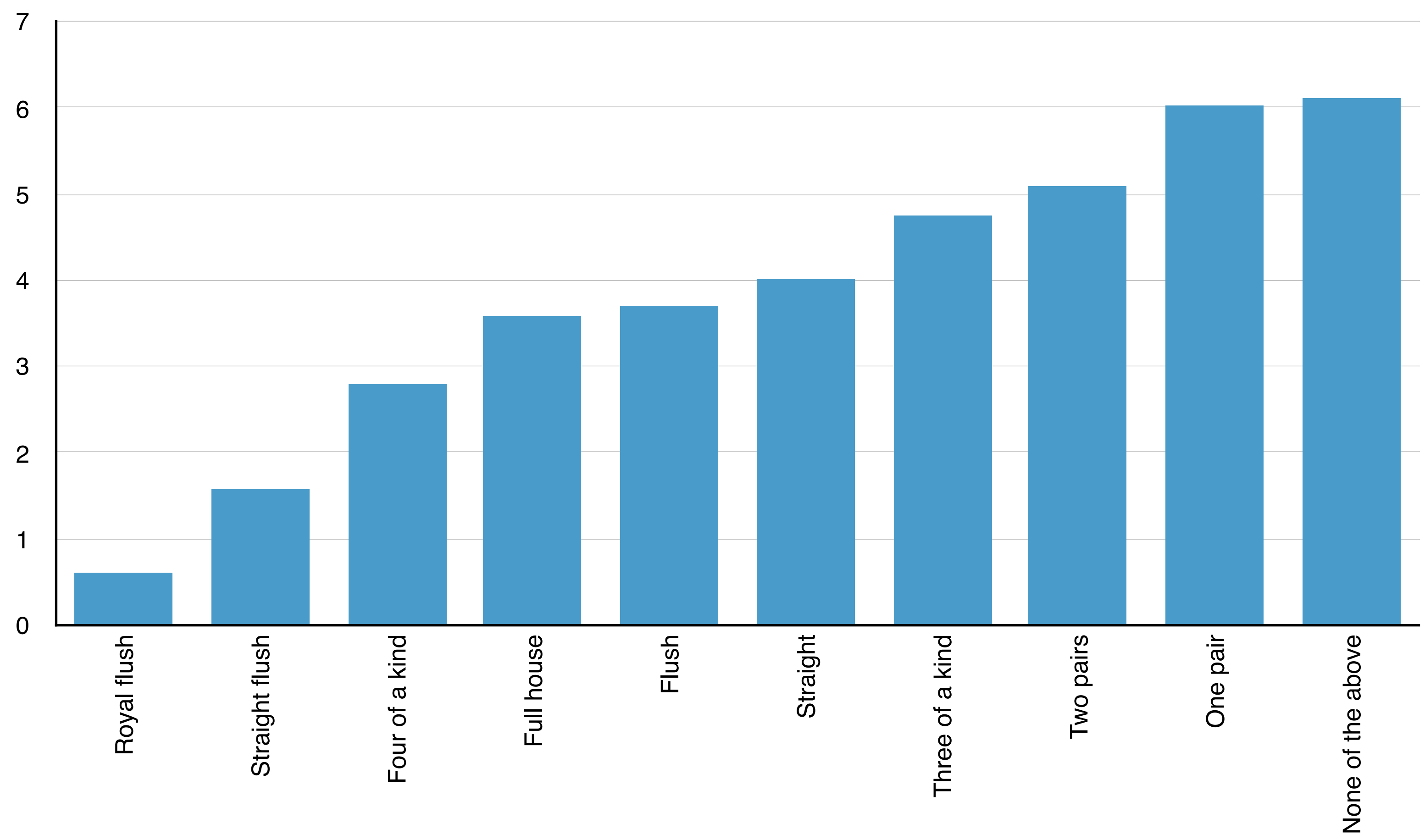
		Number of hands	Probability
Royal flush		4	0.00015%
Straight flush		36	0.00139%
Four of a kind		624	0.02401%
Full house		3744	0.14406%
Flush		5108	0.19654%
Straight		10200	0.39246%
Three of a kind		54912	2.11285%
Two pairs		123552	4.75390%
One pair		1098240	42.25690%
None of the above		1302540	50.11774%

Plot: Number of hands of each category



The top hands are not visible in this plot

Logarithmic plot: $\log_{10}$(number of hands)



Another example

How many hands don't contain *any* pair?

Another example

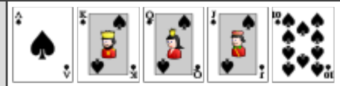
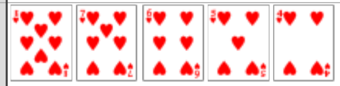
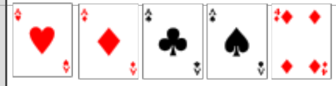
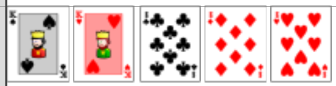
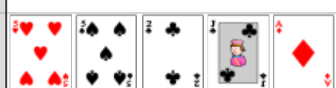
How many hands don't contain *any* pair?

Method 1

Another example

How many hands don't contain *any* pair?

Method 1

		Number of hands
Royal flush		4
Straight flush		36
Four of a kind		Contains pair
Full house		Contains pair
Flush		5108
Straight		10200
Three of a kind		Contains pair
Two pairs		Contains pair
One pair		Contains pair
None of the above		1302540
TOTAL		**1'317'888**

Another example

How many hands don't contain *any* pair?

Method 1

Method 2

		Number of hands
Royal flush		4
Straight flush		36
Four of a kind		Contains pair
Full house		Contains pair
Flush		5108
Straight		10200
Three of a kind		Contains pair
Two pairs		Contains pair
One pair		Contains pair
None of the above		1302540
TOTAL		**1’317’888**

Another example

How many hands don't contain *any* pair?

Method 1

		Number of hands
Royal flush		4
Straight flush		36
Four of a kind		Contains pair
Full house		Contains pair
Flush		5108
Straight		10200
Three of a kind		Contains pair
Two pairs		Contains pair
One pair		Contains pair
None of the above		1302540
TOTAL		**1'317'888**

Method 2

possibilities 1st card, # possibilities 2nd card, # possibilities 3rd card, # possibilities 4th card, # possibilities 5th card

$$\frac{52*48*44*40*36}{5!} = \mathbf{1'317'888}$$

Permutations don't matter

Combinations with Replacement

Choosing *r* elements from a set of *n*-elements

		Without replacement	With replacement
Ordering matters	Permutation	$P(n,r)$	n^r
Ordering doesn't matter	Combination	$C(n,r)$	**?**

Choosing *r* elements from a set of *n*-elements

		Without replacement	With replacement
Ordering matters	Permutation	$P(n,r)$	n^r
Ordering doesn't matter	Combination	$C(n,r)$	**?**

Let's call this number $A(n,r)$

Combination with replacement: example

- We have three ice cream flavors: chocolate, banana, vanilla
- How many different sets of 5-scoops can we make? **A(3,5)**

Combination with replacement: example

- We have three ice cream flavors: chocolate, banana, vanilla
- How many different sets of 5-scoops can we make? **A(3,5)**

At least one chocolate

Combination with replacement: example

- We have three ice cream flavors: chocolate, banana, vanilla
- How many different sets of 5-scoops can we make? **A(3,5)**

At least one chocolate

chocolate	banana	vanilla
5	0	0
4	1	0
4	0	1
3	2	0
3	1	1
3	0	2
2	3	0
2	2	1
2	1	2
2	0	3
1	4	0
1	3	1
1	2	2
1	1	3
1	0	4

Combination with replacement: example

- We have three ice cream flavors: chocolate, banana, vanilla
- How many different sets of 5-scoops can we make? **A(3,5)**

At least one chocolate

chocolate	banana	vanilla
5	0	0
4	1	0
4	0	1
3	2	0
3	1	1
3	0	2
2	3	0
2	2	1
2	1	2
2	0	3
1	4	0
1	3	1
1	2	2
1	1	3
1	0	4

No chocolate

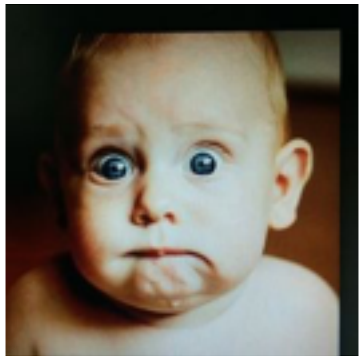
Combination with replacement: example

- We have three ice cream flavors: chocolate, banana, vanilla
- How many different sets of 5-scoops can we make? **A(3,5)**

At least one chocolate

chocolate	banana	vanilla
5	0	0
4	1	0
4	0	1
3	2	0
3	1	1
3	0	2
2	3	0
2	2	1
2	1	2
2	0	3
1	4	0
1	3	1
1	2	2
1	1	3
1	0	4

No chocolate



Combination with replacement: example

- We have three ice cream flavors: chocolate, banana, vanilla
- How many different sets of 5-scoops can we make? **A(3,5)**

At least one chocolate

chocolate	banana	vanilla
5	0	0
4	1	0
4	0	1
3	2	0
3	1	1
3	0	2
2	3	0
2	2	1
2	1	2
2	0	3
1	4	0
1	3	1
1	2	2
1	1	3
1	0	4

No chocolate

chocolate	banana	vanilla
0	5	0
0	4	1
0	3	2
0	2	3
0	1	4
0	0	5

Combination with replacement: example

- We have three ice cream flavors: chocolate, banana, vanilla
- How many different sets of 5-scoops can we make? **A(3,5)**

At least one chocolate

chocolate	banana	vanilla
5	0	0
4	1	0
4	0	1
3	2	0
3	1	1
3	0	2
2	3	0
2	2	1
2	1	2
2	0	3
1	4	0
1	3	1
1	2	2
1	1	3
1	0	4

No chocolate

chocolate	banana	vanilla
0	5	0
0	4	1
0	3	2
0	2	3
0	1	4
0	0	5

21 choices total, so $A(3,5) = 21$
but what is $A(n,r)$ in general?

At least one chocolate

At least one chocolate

chocolate	banana	vanilla
5	0	0
4	1	0
4	0	1
3	2	0
3	1	1
3	0	2
2	3	0
2	2	1
2	1	2
2	0	3
1	4	0
1	3	1
1	2	2
1	1	3
1	0	4

At least one chocolate

At least one chocolate

chocolate	banana	vanilla
5	0	0
4	1	0
4	0	1
3	2	0
3	1	1
3	0	2
2	3	0
2	2	1
2	1	2
2	0	3
1	4	0
1	3	1
1	2	2
1	1	3
1	0	4

Take out that chocolate scoop
Same as number of 4-scoops

At least one chocolate

At least one chocolate

chocolate	banana	vanilla
5	0	0
4	1	0
4	0	1
3	2	0
3	1	1
3	0	2
2	3	0
2	2	1
2	1	2
2	0	3
1	4	0
1	3	1
1	2	2
1	1	3
1	0	4

=

Take out that chocolate scoop
Same as number of 4-scoops

chocolate	banana	vanilla
4	0	0
3	1	0
3	0	1
2	2	0
2	1	1
2	0	2
1	3	0
1	2	1
1	1	2
1	0	3
0	4	0
0	3	1
0	2	2
0	1	3
0	0	4

This is number of 4-scoops of three flavors, so $A(3,4)$

No chocolate

No chocolate

chocolate	banana	vanilla
0	5	0
0	4	1
0	3	2
0	2	3
0	1	4
0	0	5

No chocolate

No chocolate

chocolate	banana	vanilla
0	5	0
0	4	1
0	3	2
0	2	3
0	1	4
0	0	5

5-scoops from two flavors only

No chocolate

No chocolate

chocolate	banana	vanilla
0	5	0
0	4	1
0	3	2
0	2	3
0	1	4
0	0	5

=

5-scoops from two flavors only

banana	vanilla
5	0
4	1
3	2
2	3
1	4
0	5

This is number of 5-scoops of two flavors, so $A(2,5)$

Recursion

$A(3,5) = A(3,4) + A(2,5)$

$A(3,4)$

chocolate	banana	vanilla
5	0	0
4	1	0
4	0	1
3	2	0
3	1	1
3	0	2
2	3	0
2	2	1
2	1	2
2	0	3
1	4	0
1	3	1
1	2	2
1	1	3
1	0	4

$A(2,5)$

chocolate	banana	vanilla
0	5	0
0	4	1
0	3	2
0	2	3
0	1	4
0	0	5

Recursion

$$A(3,5) = A(3,4) + A(2,5)$$

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$$A(n,r) = A(n,r\text{-}1) + A(n\text{-}1,r) \quad \text{for } r,\ n \geq 1$$

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Combinations containing at least one copy of the first element of original set

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Combinations containing at least one copy of the first element of original set

Combinations containing no copy of the first element of original set

Trivial cases

$A(n, 1) = n$ for $n \geq 1$

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If you have n ice cream flavors, but only one scoop, you can only have n different ice cream servings.

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$A(1, r) = 1$ for $r \geq 1$

Trivial cases

$A(n, 1) = n$ for $n \geq 1$

If you have *n* ice cream flavors, but only one scoop, you can only have *n* different ice cream servings.

$A(1, r) = 1$ for $r \geq 1$

If you have only one ice cream flavor, then no matter how many scoops you take, you will end up having only one type of ice cream serving.

$A(n,r) = A(n,r-1) + A(n-1,r)$

	1	2	3	4	5
1					
2					
3					
4					
5					

$A(n,r) = A(n,r-1) + A(n-1,r)$

	1	2	3	4	5
1	1	2	3	4	5
2					
3					
4					
5					

$A(n,1) = n$ for $n \geq 1$

$A(n,r) = A(n,r-1) + A(n-1,r)$

	1	2	3	4	5
1	1	2	3	4	5
2	1				
3	1				
4	1				
5	1				

$A(n,1) = n$ for $n \geq 1$

$A(1,r) = 1$ for $r \geq 1$

$A(n,r) = A(n,r-1) + A(n-1,r)$

	1	2	3	4	5
1	1	2	3	4	5
2	1				
3	1				
4	1				
5	1				

$A(n,1) = n$ for $n \geq 1$

$A(1,r) = 1$ for $r \geq 1$

$A(n,r) = A(n,r-1) + A(n-1,r)$

	1	2	3	4	5
1	1	2	3	4	5
2	1	3			
3	1				
4	1				
5	1				

add

$A(n,1) = n$ for $n \geq 1$

$A(1,r) = 1$ for $r \geq 1$

$A(n,r) = A(n,r-1) + A(n-1,r)$

	1	2	3	4	5
1	1	2	3	4	5
2	1	3	6		
3	1				
4	1				
5	1				

add

add

$A(n,1) = n$ for $n \geq 1$

$A(1,r) = 1$ for $r \geq 1$

$A(n,r) = A(n,r-1) + A(n-1,r)$

	1	2	3	4	5
1	1	2	3	4	5
2	1	3	6	10	
3	1				
4	1				
5	1				

$A(n,1) = n$ for $n \geq 1$

$A(1,r) = 1$ for $r \geq 1$

$A(n,r) = A(n,r-1) + A(n-1,r)$

	1	2	3	4	5
1	1	2	3	4	5
2	1	3	6	10	15
3	1				
4	1				
5	1				

add add add add

$A(n,1) = n$ for $n \geq 1$

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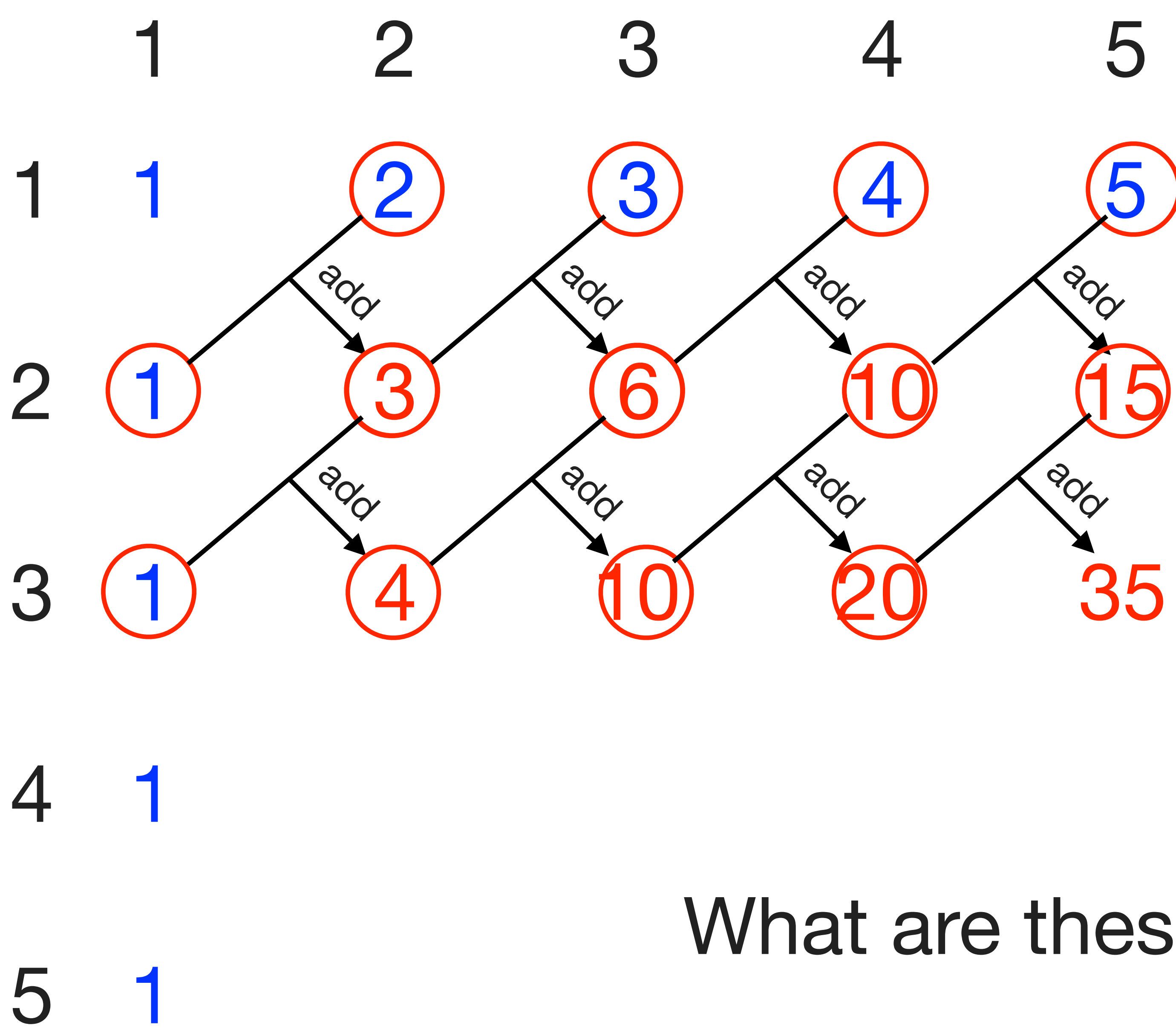

$A(n,1) = n$ for $n \geq 1$

$A(1,r) = 1$ for $r \geq 1$

What are these numbers?

Intuition on what $A(3,5)$ could be

Intuition on what *A*(3,5) could be

Intuition on what *A*(3,5) could be

These numbers are ≥ 0
Their sum is the
number of scoops = 5

2 0 3

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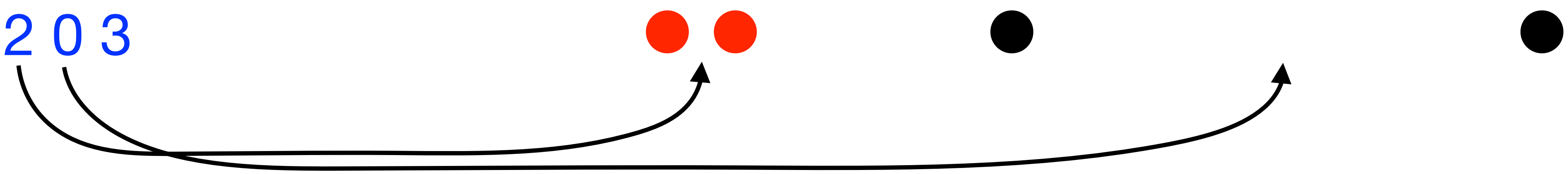

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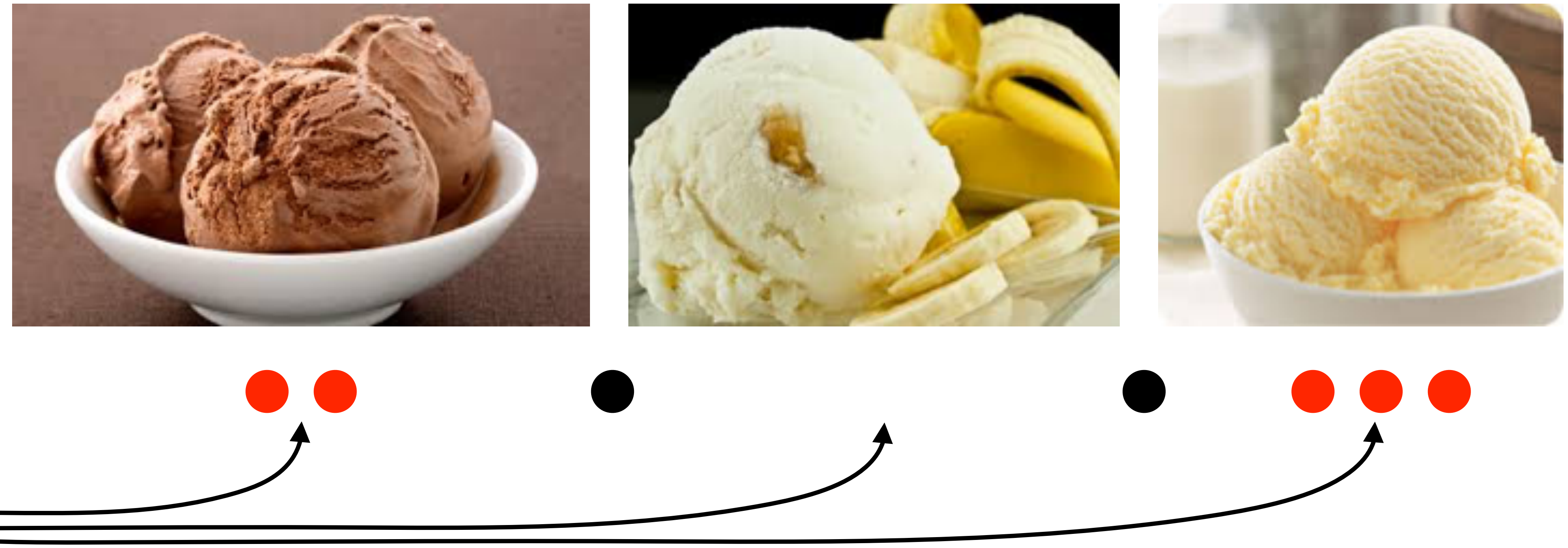

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Intuition on what *A*(3,5) could be

These numbers are ≥ 0
Their sum is the
number of scoops = 5

2 0 3 → ● ● | ● | ● ● ●

1 3 1 →

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These numbers are ≥ 0
Their sum is the
number of scoops = 5

2 0 3

1 3 1

Intuition on what *A*(3,5) could be

These numbers are ≥ 0
Their sum is the
number of scoops = 5

2 0 3 → ● ● | | ● ● ●

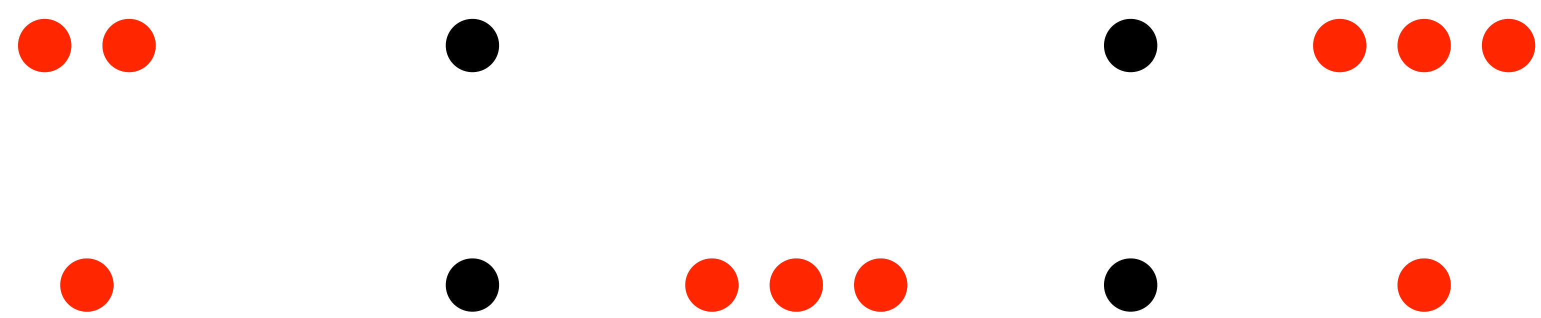
1 3 1 → ● | ● ● ● | ●

Intuition on what *A*(3,5) could be



These numbers are ≥ 0
Their sum is the
number of scoops = 5

2 0 3 →
1 3 1 →



0 5 0 →

Intuition on what *A*(3,5) could be



These numbers are ≥ 0
Their sum is the
number of scoops = 5

2 0 3

1 3 1

0 5 0

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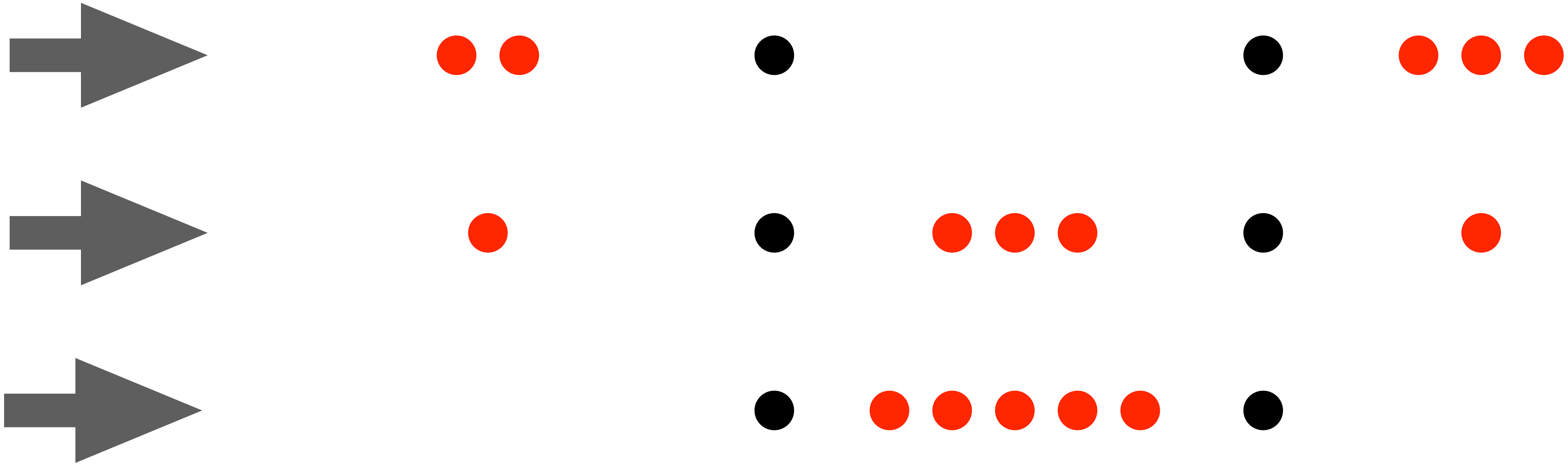
2 0 3

1 3 1

0 5 0



Possibilities are mapped to sequences of 7 black and red marbles of which exactly 5 are red (and exactly 2 are black)

Intuition on what *A*(3,5) could be

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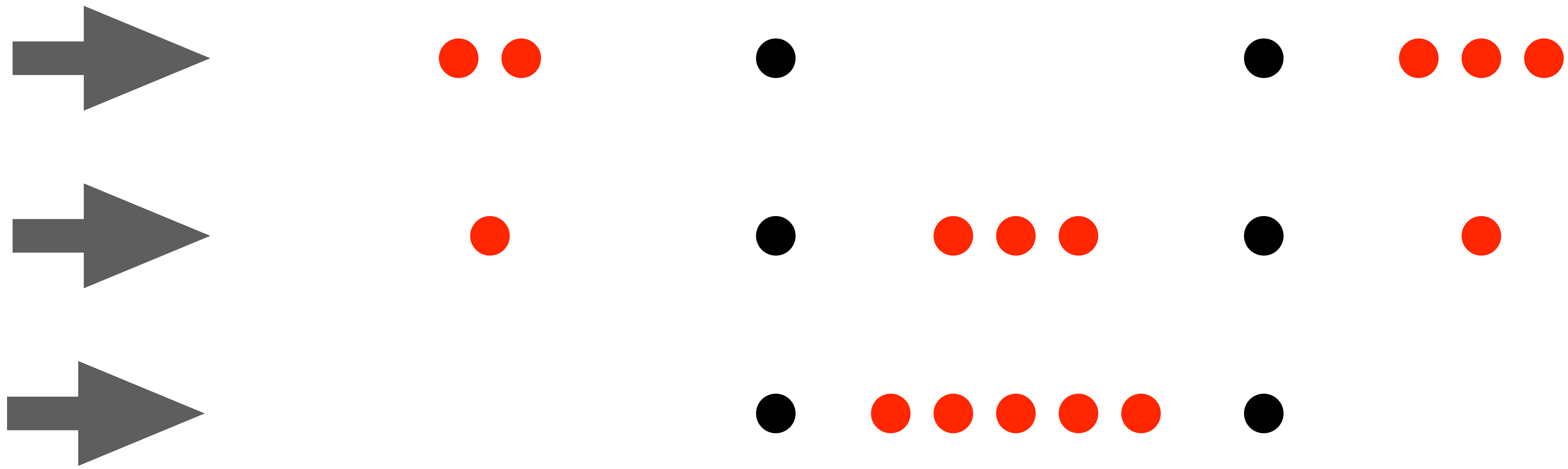
2 0 3

1 3 1

0 5 0



Possibilities are mapped to sequences of 7 black and red marbles of which exactly 5 are red (and exactly 2 are black)

C(7,5) = 7!/(2!*5!) = 7*3=21

Intuition on what *A*(*n*,*r*) could be

Item 1	Item 2	Item 3		Item *n*-1	Item *n*

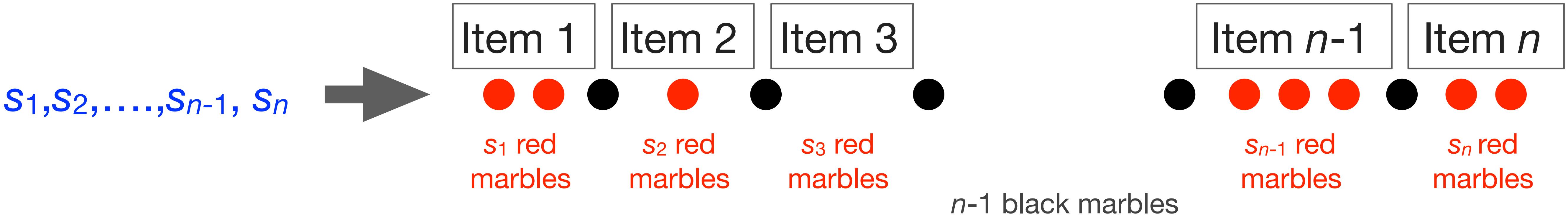
Intuition on what $A(n,r)$ could be

We first put in n-1 markers

Intuition on what $A(n,r)$ could be

n-1 black marbles

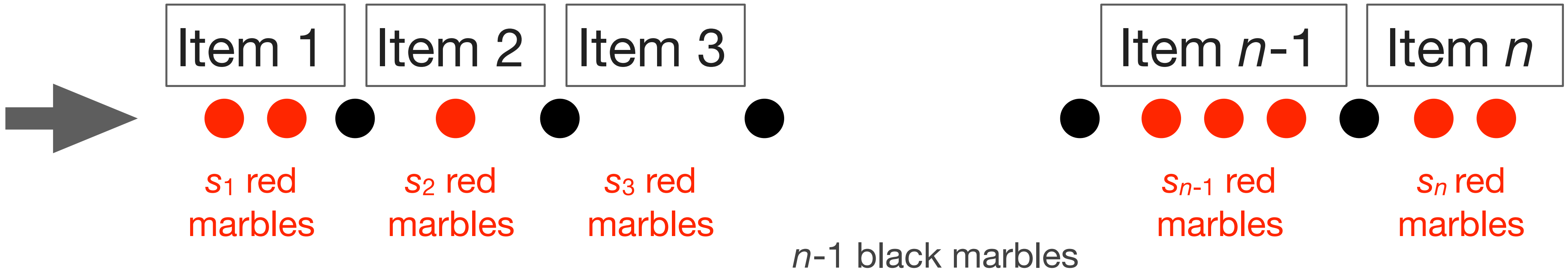
Intuition on what $A(n,r)$ could be



Intuition on what $A(n,r)$ could be

$s_1, s_2, \ldots, s_{n-1}, s_n$

These numbers are ≥ 0
Their sum is r



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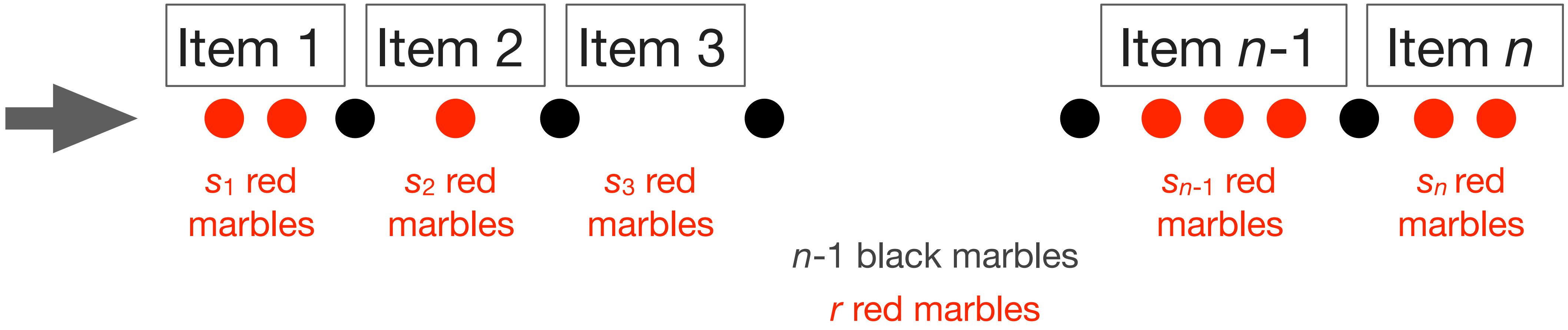

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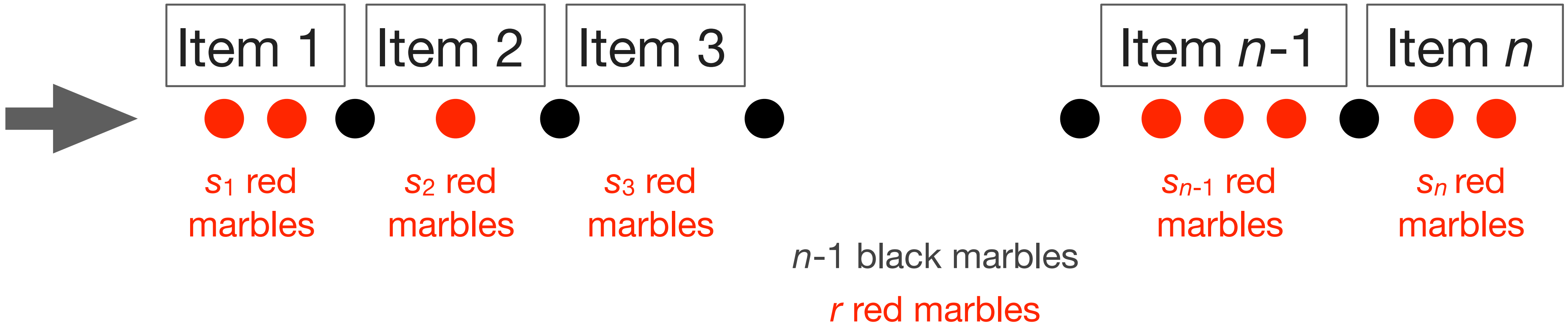

Total number of marbles is $n - 1 + r$

Intuition on what $A(n,r)$ could be

$s_1, s_2, \ldots, s_{n-1}, s_n$

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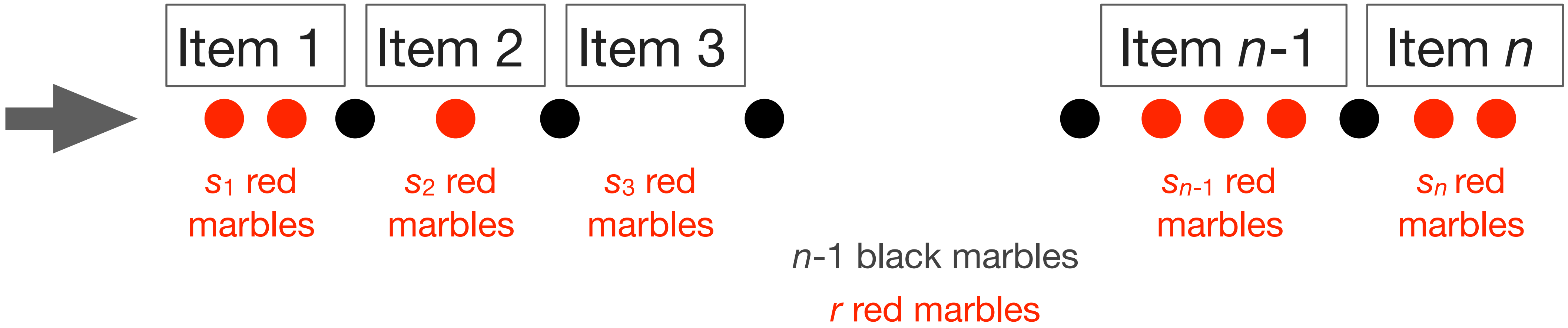
Their sum is r



Total number of marbles is $n - 1 + r$

Possibilities are mapped to sequences of n-1+r black and red marbles of which exactly r are red (and exactly n-1 are black)

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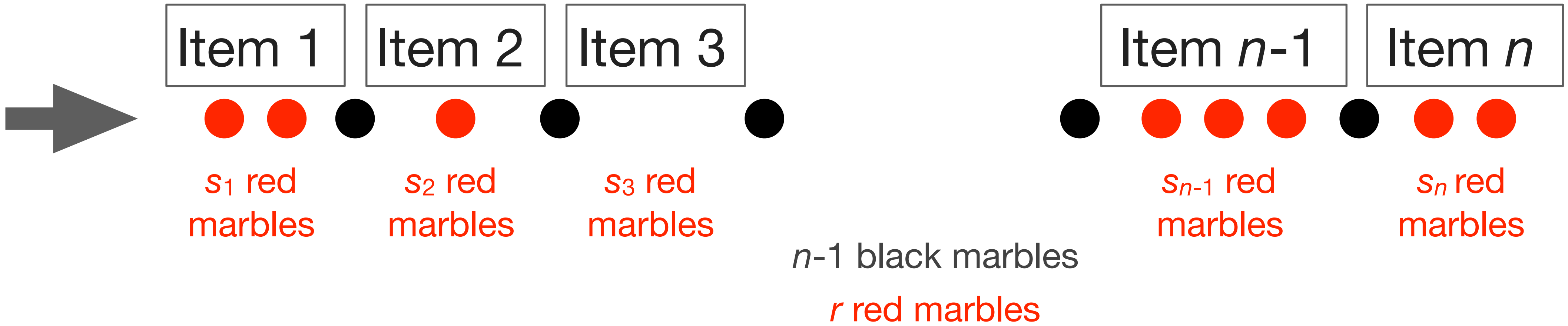

Total number of marbles is $n - 1 + r$

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C(n-1+r, r)

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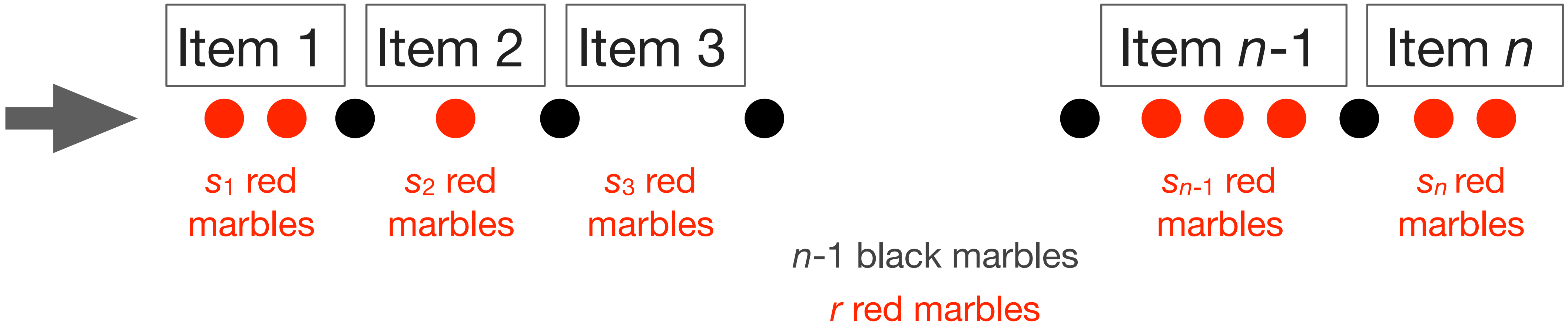

Total number of marbles is $n - 1 + r$

Possibilities are mapped to sequences of n-1+r black and red marbles of which exactly r are red (and exactly n-1 are black)

$C(n-1+r, r)$

$A(n,r) = C(n-1+r,r)$?

Proof by double induction

$P(n)$ = "for all $r \geq 1$ we have $A(n,r) = C(n-1+r,r)$". Need to prove $P(n)$ for all $n \geq 1$.

Induction basis: $n=1$; $A(1,r) = 1$ for $r \geq 1$ (if only one object, then only r-combination is repetition of that object r times).
On the other hand, $C(n-1+r, r) = C(r,r) = 1$, so correct.

Proof by double induction

$P(n)$ = “for all $r \geq 1$ we have $A(n,r) = C(n\text{-}1+r,r)$”. Need to prove $P(n)$ for all $n \geq 1$.

Induction basis: ✓

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Induction step: Assume $P(n)$, prove $P(n+1)$:

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Induction step: Assume $P(n)$, prove $P(n+1)$:
$Q(r)$ = "We have $A(n+1,r) = C(n+r,r)$"

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Prove using induction on r.

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Prove using induction on r.

Induction basis: $r=1$; $A(n+1,1) = n+1$ (trivial case);

On the other hand, $C(n+1-1+1,1) = n+1$.

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Prove using induction on r.

Induction basis: ✓

Induction step: Assume $Q(r)$, prove $Q(r+1)$: Use recursion.

Proof by double induction

$P(n)$ = "for all $r \geq 1$ we have $A(n,r) = C(n\text{-}1+r,r)$". Need to prove $P(n)$ for all $n \geq 1$.

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Prove using induction on r.

Induction basis: ✓

$A(n,r) = A(n,r\text{-}1) + A(n\text{-}1,r)$

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Induction step: Assume $Q(r)$, prove $Q(r+1)$: Use recursion.

$A(n+1,r+1) = A(n+1,r) + A(n,r+1)$

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Induction step: Assume $Q(r)$, prove $Q(r+1)$: Use recursion.

$A(n+1,r+1) = A(n+1,r) + A(n,r+1)$

$Q(r)$ ↓ ↓ $P(n)$

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$$A(n+1,r+1) = A(n+1,r) + A(n,r+1)$$

$Q(r)$ ↓ ↓ $P(n)$

$$= C(n+r,r) + C(n+r,r+1)$$

A little theorem (Pascal's Identity)

$$\forall n, k \in \mathbb{N}\colon C(n,k) + C(n,k-1) = C(n+1,k)$$

$$C(n,k) + C(n,k-1) \quad = \quad \binom{n}{k} + \binom{n}{k-1}$$

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$$\begin{aligned} C(n,k) + C(n,k-1) &= \binom{n}{k} + \binom{n}{k-1} \\ &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \end{aligned}$$

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&= \frac{n!}{(k-1)!(n-k)!}\,\frac{n+1}{k(n-k+1)} \\
&= \frac{(n+1)!}{k!(n-k+1)!}
\end{aligned}$$

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&= \frac{(n+1)!}{k!(n-k+1)!} \\
&= C(n+1,k)
\end{aligned}$$

Proof by double induction

$P(n)$ = "for all $r \geq 1$ we have $A(n,r) = C(n-1+r,r)$". Need to prove $P(n)$ for all $n \geq 1$.

Induction basis: ✓

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$Q(r)$ ↓ ↓ $P(n)$

$= C(n+r,r) + C(n+r,r+1)$

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$P(n)$ = "for all $r \geq 1$ we have $A(n,r) = C(n-1+r,r)$". Need to prove $P(n)$ for all $n \geq 1$.

<u>Induction basis</u>: ✓

<u>Induction step</u>: Assume $P(n)$, prove $P(n+1)$:

$Q(r)$ = "We have $A(n+1,r) = C(n+r,r)$"

Prove using induction on r.

<u>Induction basis</u>: ✓

$A(n,r) = A(n,r-1) + A(n-1,r)$

<u>Induction step</u>: Assume $Q(r)$, prove $Q(r+1)$: Use recursion.

$A(n+1,r+1) = A(n+1,r) + A(n,r+1)$

$Q(r)$ ↓ ↓ $P(n)$

$= C(n+r,r) + C(n+r,r+1)$

$C(n,k)+C(n,k-1)=C(n+1,k)$

$= C(n+r+1, r+1)$

Proof by double induction

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$A(n+1,r+1) = A(n+1,r) + A(n,r+1)$

$Q(r)$ ↓ ↓ $P(n)$

$= C(n+r,r) + C(n+r,r+1)$

$C(n,k)+C(n,k-1)=C(n+1,k)$

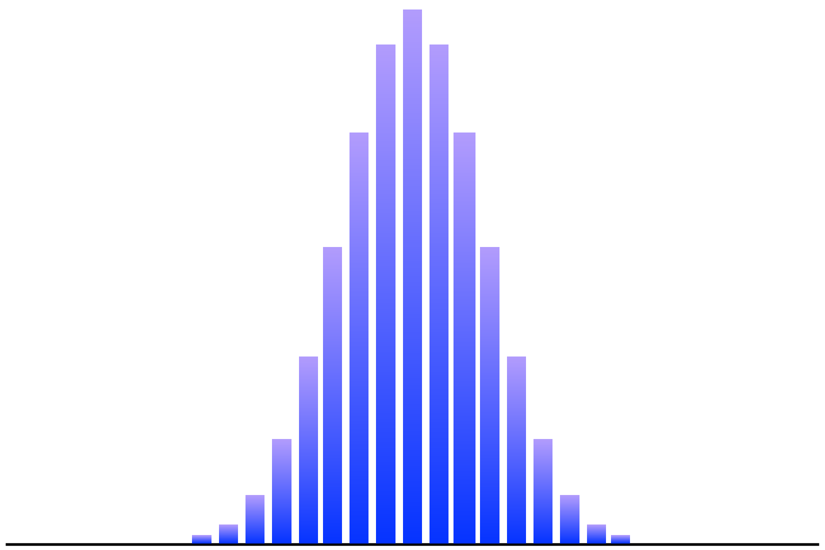
$= C(n+r+1, r+1)$

$= C((n+1)-1+(r+1), r+1)$ ✓

Choosing *r* elements from a set of *n*-elements

		Without replacement	With replacement
Ordering matters	Permutation	$P(n,r)$	n^r
Ordering doesn't matter	Combination	$C(n,r)$	$\mathbf{C(n+r-1,r)}$

More Binomials



Binomial Theorem & combinatorial proof

$$\forall n \geq 1 \colon (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

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Look at an example: What is the coefficient of x^3 in $(1+x)^5$?
In how many ways do we get x^3 in $(1+x)$* $(1+x)$* $(1+x)$* $(1+x)$* $(1+x)$?

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= Number of ways to pick 3 terms among 5, without replacement, ordering irrelevant = $C(5,3)$

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....

= Number of ways to pick 3 terms among 5, without replacement, ordering irrelevant = *C*(5,3)

In general: number of ways to pick *k* terms among *n*, so = *C*(*n*,*k*) = $\binom{n}{k}$

Binomial Theorem & algebraic proof

pp. 403-409

$$\forall n \geq 1 \colon (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Binomial Theorem & algebraic proof

pp. 403-409

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Use induction on *n*.

More theorems

$$\forall n \geq 1\colon (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

More theorems

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

pp. 405-406

$$\forall n \geq 1 : (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

More theorems

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pp. 405-406

Proof: Plug x=1 into the binomial theorem.

More theorems

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pp. 405-406

Proof: Plug x=1 into the binomial theorem.

Combinatorial proof: Count number of bit-strings of length n as sum of number of bit-strings of length n which have exactly k ones, k from 0 to n.

Vandermonde identity

Vandermonde identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k}, \quad \text{if } r < m, n$$

pp. 408-409

Vandermonde identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}, \qquad \text{if } r < m, n$$

pp. 408-409

Combinatorial proof: Subset of size *r* of {1,2,…,*m*+*n*} is obtained from all combinations of

- subsets of size *k* of {1,2,…,*n*} and
- all subsets of size *r*-*k* of {*n*+1,…,*m*+*n*},
- for *k* from 0 to *r*.

Vandermonde identity

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pp. 408-409

Combinatorial proof: Subset of size r of $\{1,2,\ldots,m+n\}$ is obtained from all combinations of

- subsets of size k of $\{1,2,\ldots,n\}$ and
- all subsets of size r-k of $\{n+1,\ldots,m+n\}$,
- for k from 0 to r.

Algebraic proof: Look at coefficient of x^r in $(1+x)^n(1+x)^m$

Consequence

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{n}{k} \binom{m}{r-k}, \qquad \text{if } r < m, n$$

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Set *m*=*n*=*r* in Vandermonde's identity