Expectation
Variance

Markov Inequality

Y random variable, $Y(s) \geq 0$, then $P(Y \geq x) \leq E(Y)/x$

Andrei Andreyevich Markov
1856 - 1922

Chebyshev Inequality

Y random variable, then $P(\,|Y-E(Y)| \geq x) \leq V(Y)/x^2$

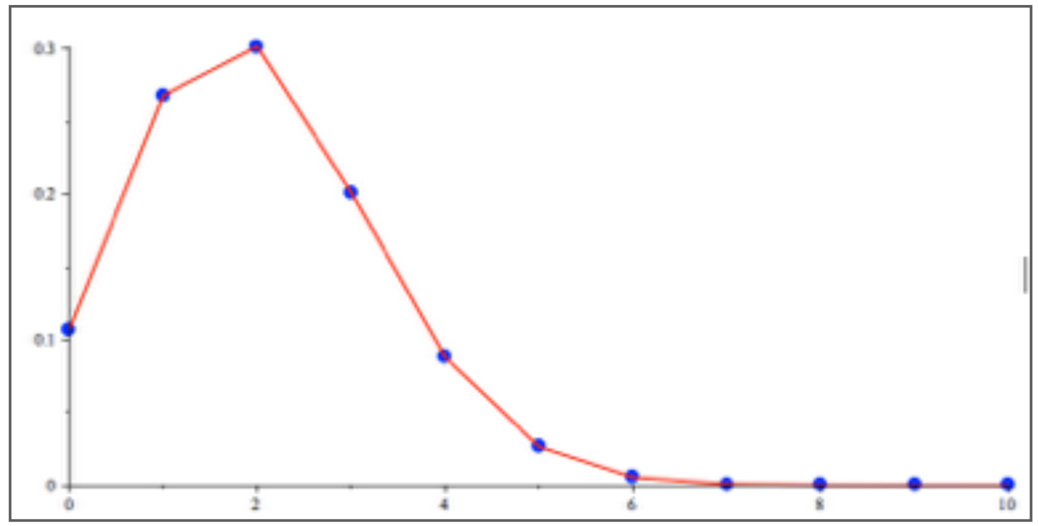
Pafnuty Lvovich Chebyshev
1821 - 1894

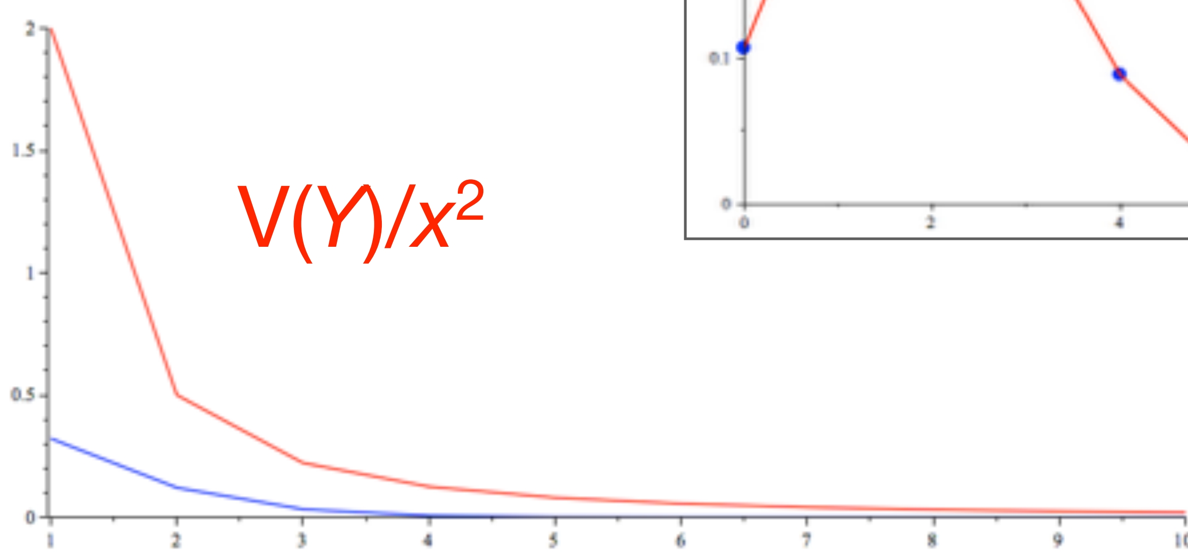
Quality of Chebyshev Inequality for Binomial Distribution

$Y = B(n,p)$ n independent Bernoulli trials with probability p

$n = 10$, $p = 0.2$

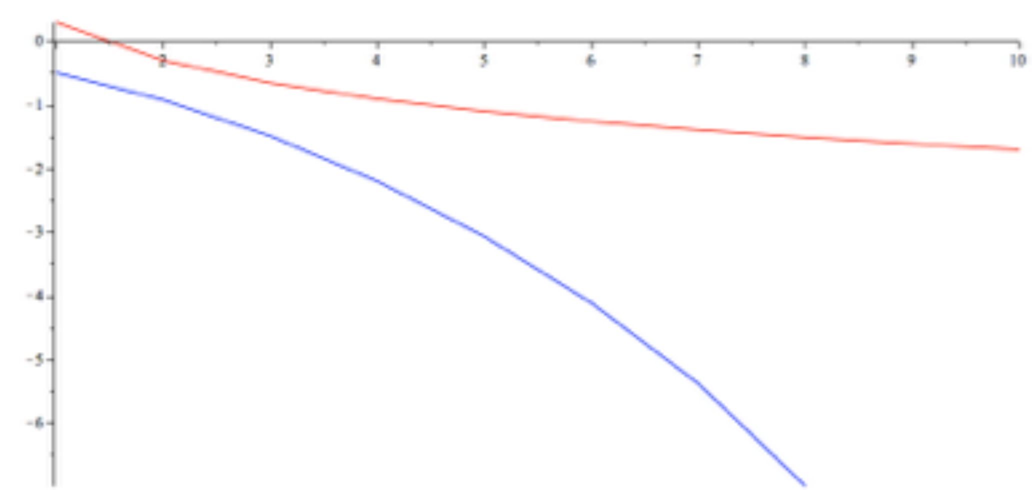
$V(Y) = n*p*(1-p) = 1.6$





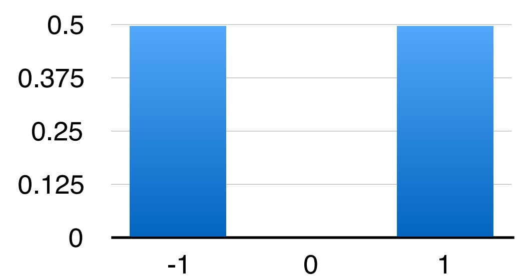
$P(\ |Y-E(Y)| \geq x)$

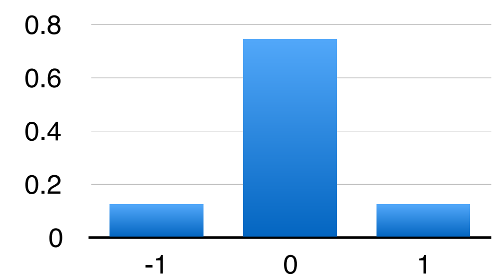
Logarithmic plot

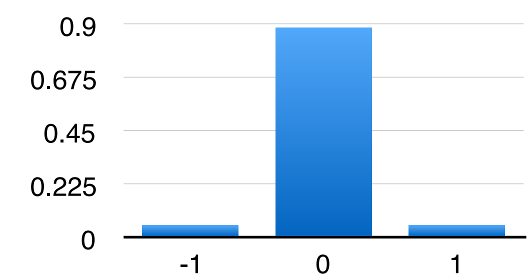


Can Chebyshev's Inequality be Improved?

- It is <u>best possible</u> for some distributions
- Without further knowledge of the probability distribution, it <u>cannot be improved</u>.
- X probability distribution on the set {-1,0,1} given by
  - $P(X=-1) = 1/(2*k^2)$
  - $P(X=0) = 1-1/k^2$
  - $P(X=1) = 1/(2*k^2)$







- $E(X) = (-1)*1/(2*k^2) + 0*(1-1/k^2)+1*1/(2*k^2) = 0$
- $V(X) = (-1)^2*1/(2*k^2) + 0^2*(1-1/k^2)+1^2*1/(2*k^2) = 1/k^2$
- $P(|X| \geq 1) = P(X=1) + P(X=-1) = 1/k^2 = V(X)/1^2$

The Monty Hall Problem

Problem Statement

- You have three doors.

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A

B

C

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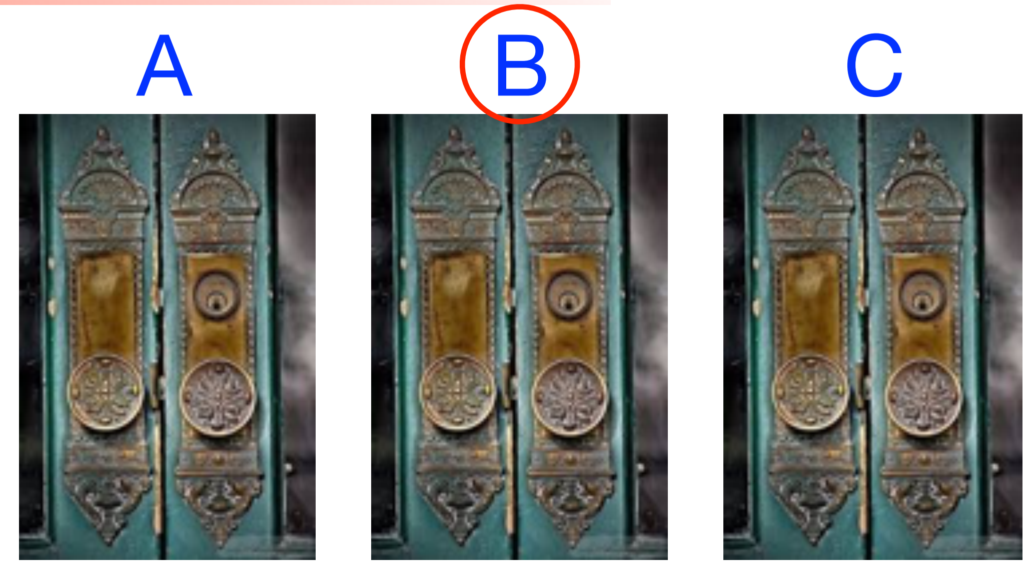

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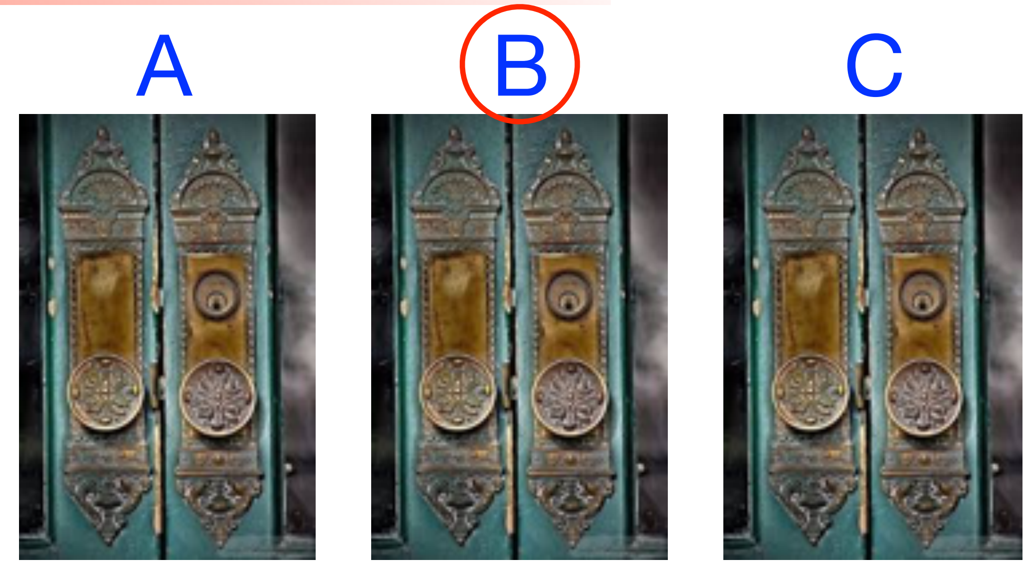
- You have three doors.
- There is a prize behind one door, and nothing behind the other two
- A contestant is asked to pick one door
- The host (Monty Hall) opens one of the two remaining doors <u>behind which there is no prize</u>
  - <u>He knows where the prize is</u>



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A B C

Better luck next time

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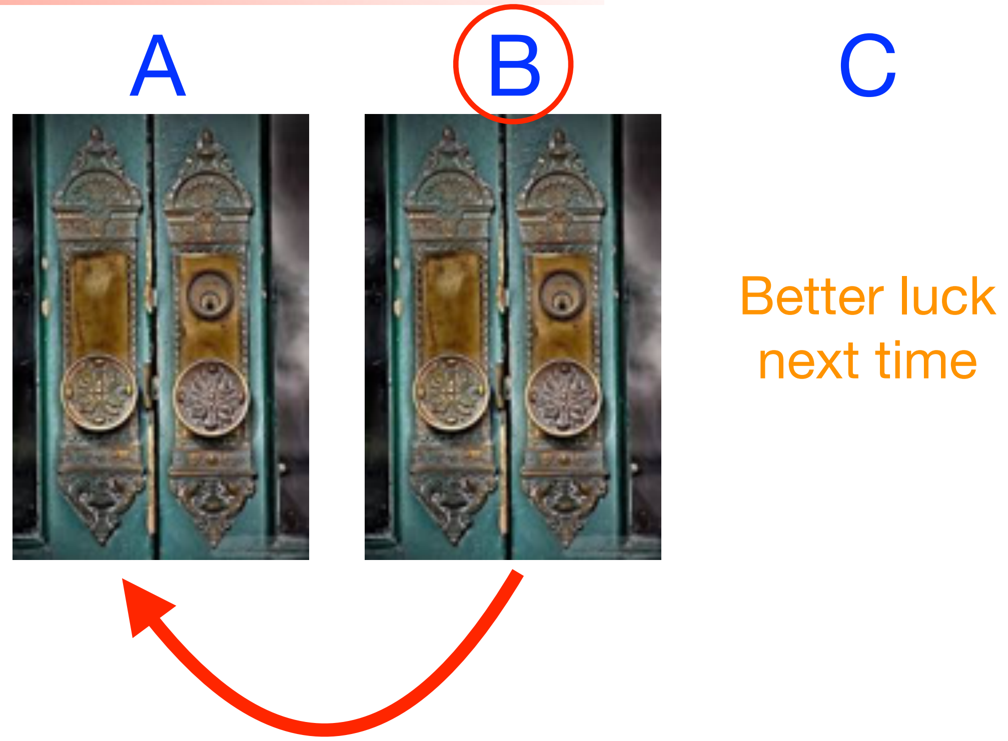
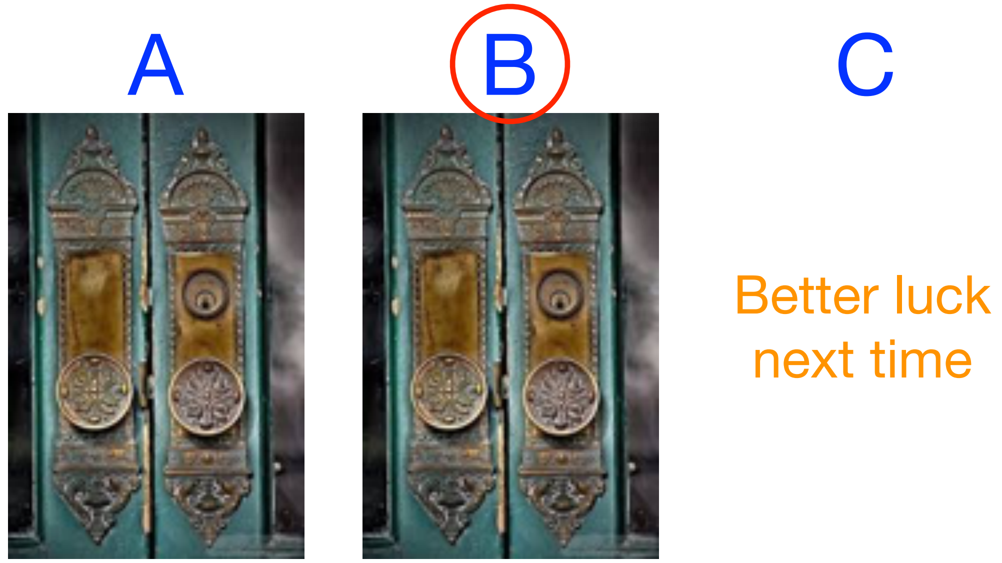


Is there any advantage?

Is there an Advantage? "Obviously Not!"

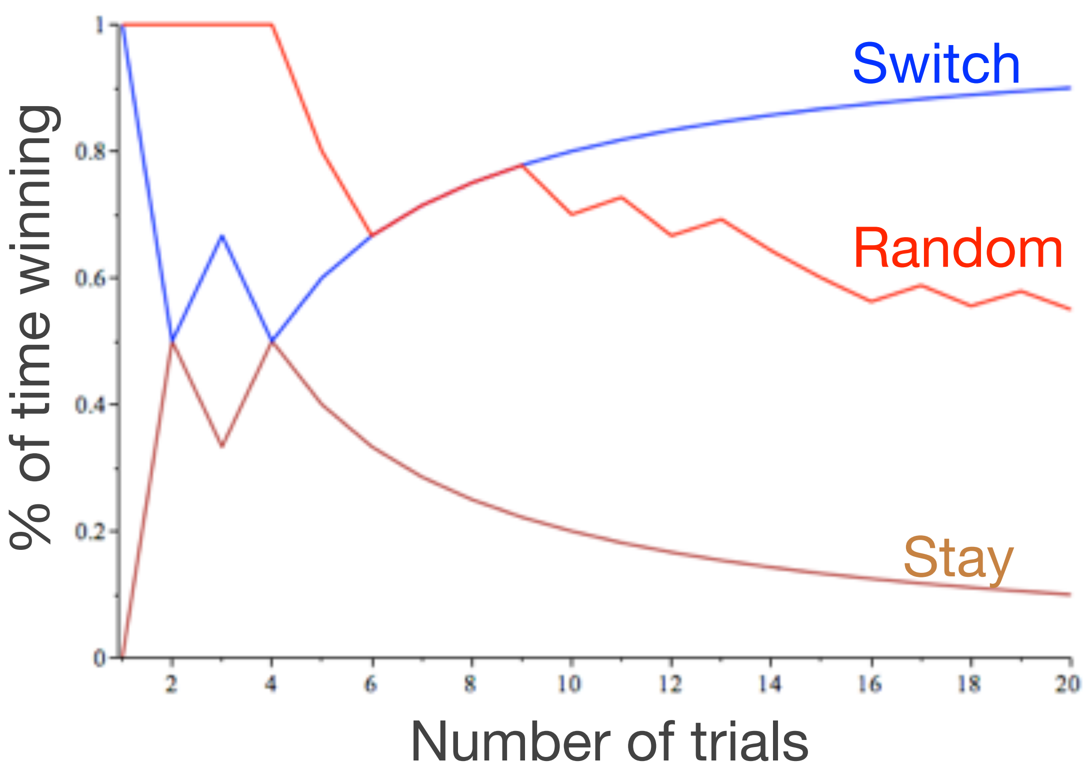
- Once Monty Hall opens one of the doors, the prize is behind one of the other two
- The position of the prize was randomly selected to begin with
- The probability of having the prize behind any of the remaining doors is 50%
- So, whether you switch or stay with your choice, you have 50% chance of winning

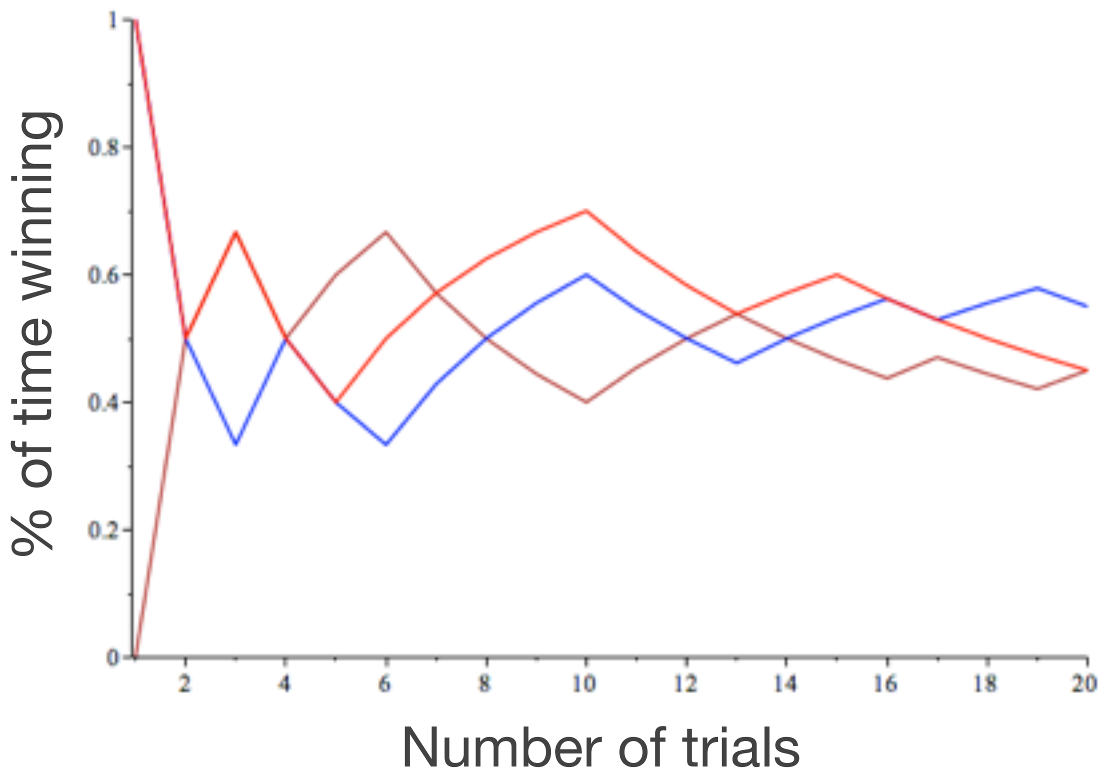


Well Then, Let's Play the Game!

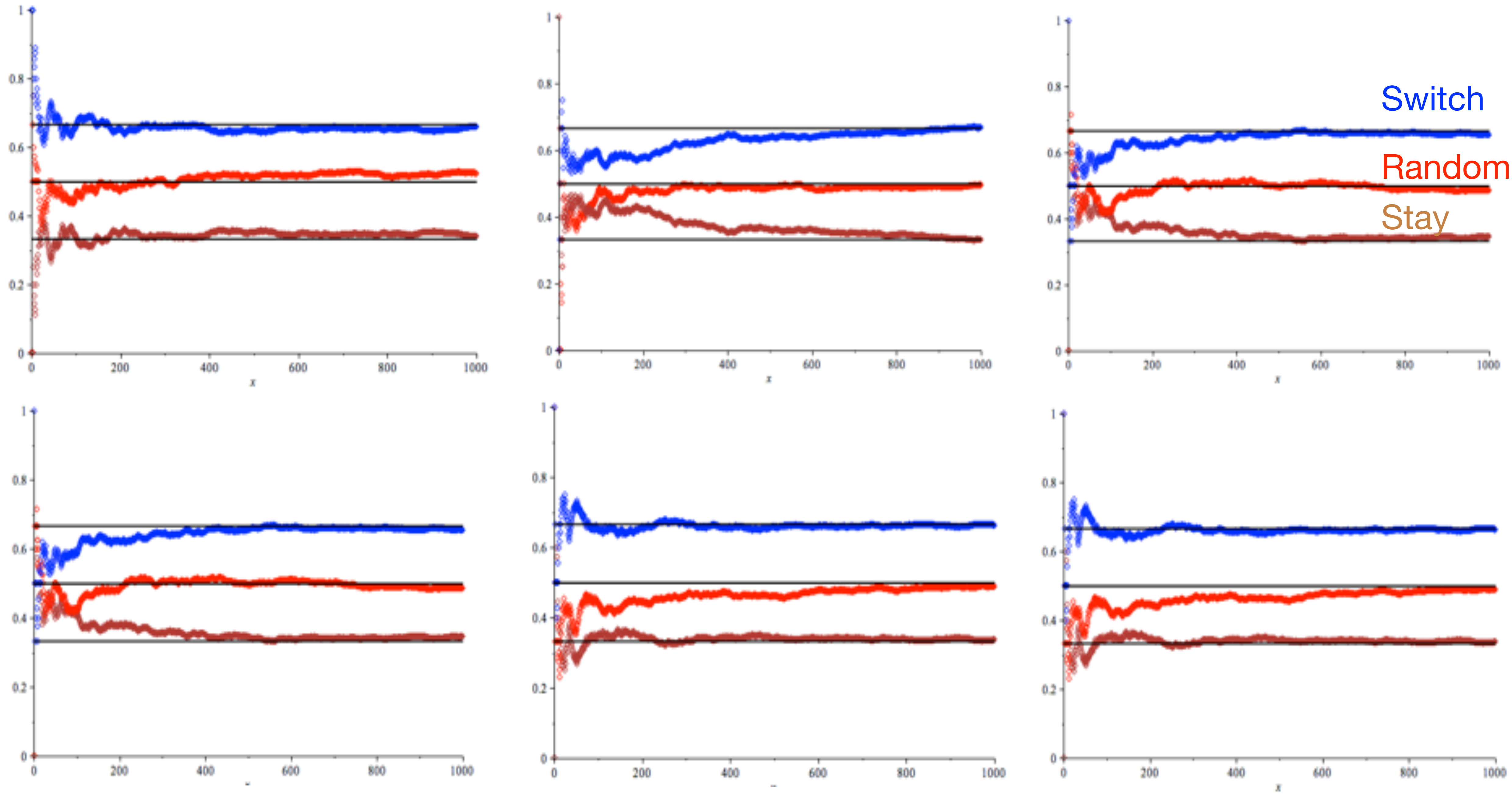
- We play it 20 times
- One person chooses the door behind which the prize is (and keeps it to herself)
- One person always switches
- One person always stays with original choice
- One person switches randomly (per coin flip)
- We record the number of times each person wins (sometimes two persons may win)

Here are Computer Simulations of 20 Games





Here are Computer Simulations of 1000 Games



Switching seems to win with probability 2/3

Random seems to win with probability 1/2

Staying seems to win with probability 1/3

Marilyn Vos Savant

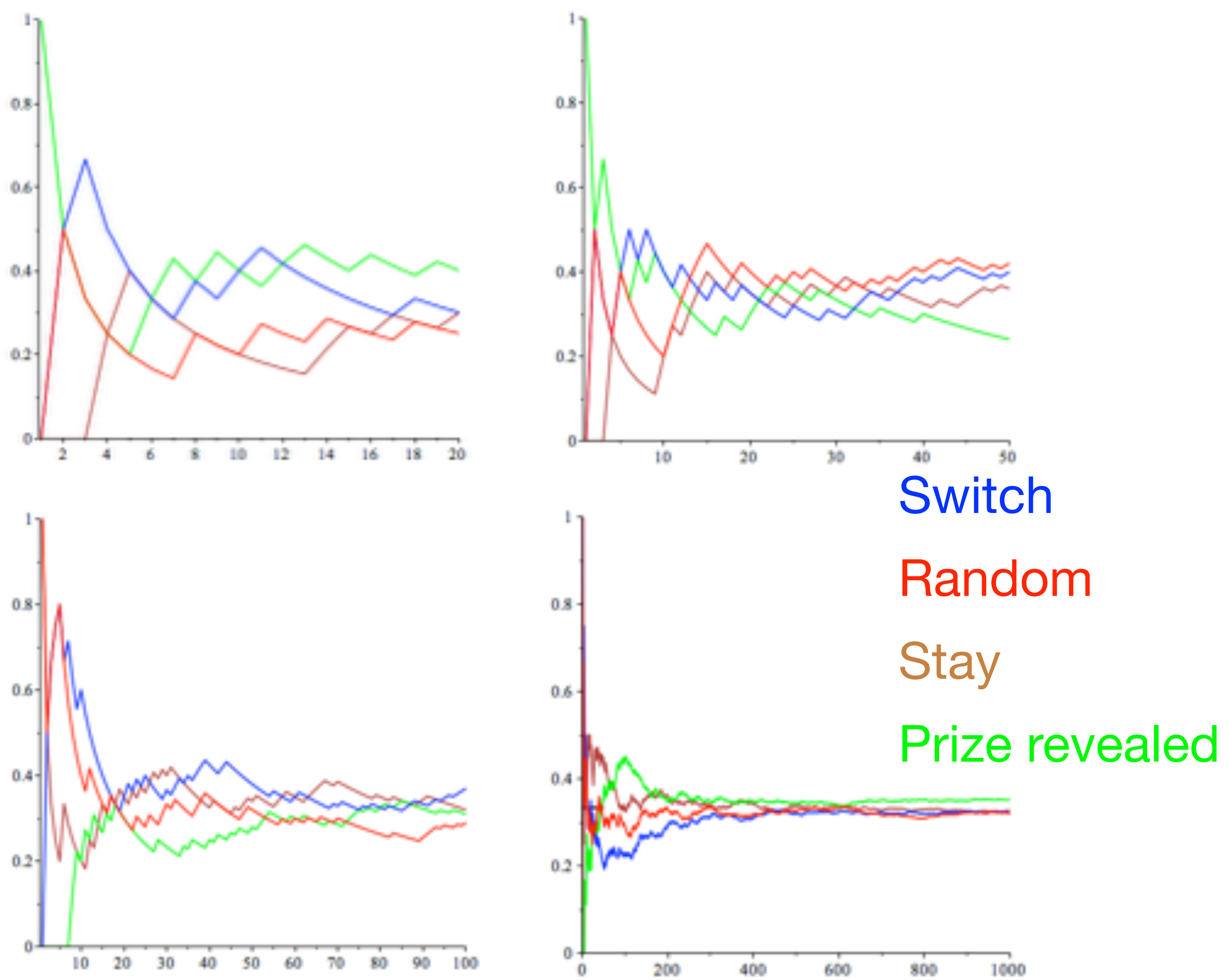
- Ran a Sunday column in the "Parade" magazine called "Ask Marilyn"
  - Solved puzzles, answered questions on various subjects
- Was asked about the Monty Hall Problem (1990)
  - Answered "If you switch, you win with probability 2/3"
- This created a storm of controversy in the US
  - Received 10000 letters objecting to her solution
  - Among them 1000 PhD's in Math and Physics
- But… She was right!

Why?

- Extremely important:
  - The host always opens a remaining door behind which there is no prize (because he knows where the prize is)
- Therefore, if you switch, the only way of losing is if the prize was behind the original door
  - The probability of that is 1/3
  - So, you win with probability 2/3 if you switch and you win with probability 1/3 if you stay
- If you randomly switch, then you switch between the probabilities 1/3 and 2/3 with equal chance, so you win with probability (1/3+2/3)/2 = 1/2.

What if the Host doesn't Know where the Prize is?

- Host randomly opens one of the other two doors
- If there is a prize behind that, the game is finished — no winner
- If there is no prize, the contestant can
  - Switch
  - Stay
  - Randomly select among the other two doors
- Is there an advantage?



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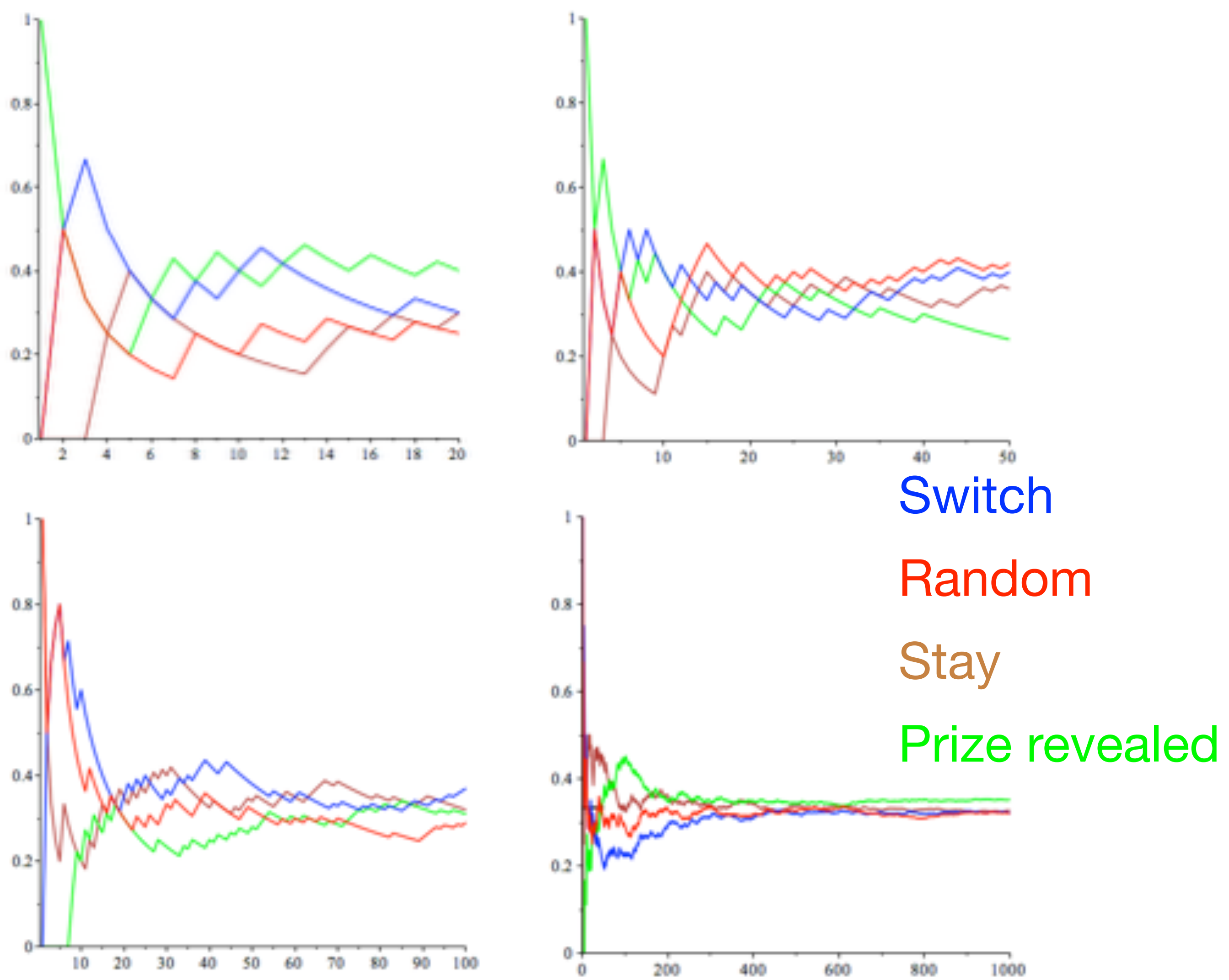

No advantage in this case