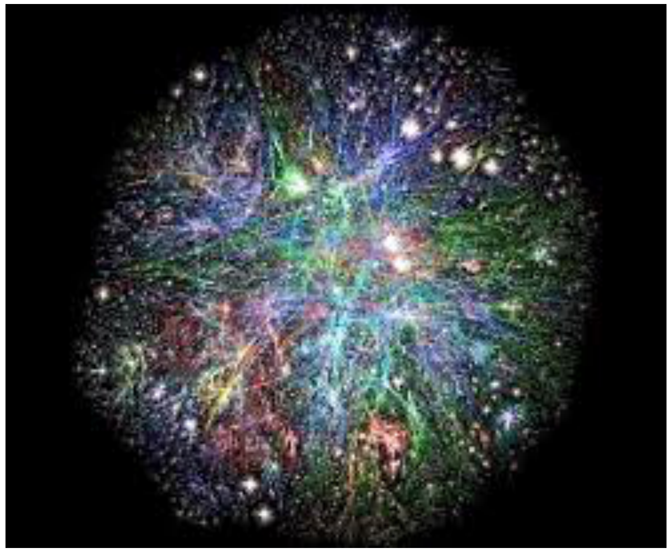
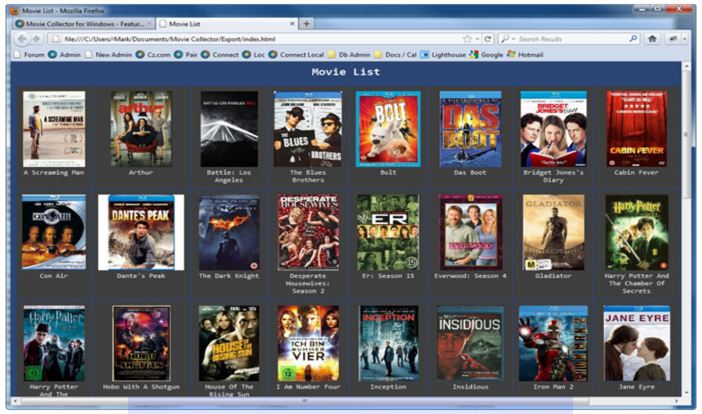
Lecture 23
Graphs



What do These Examples Have in Common?

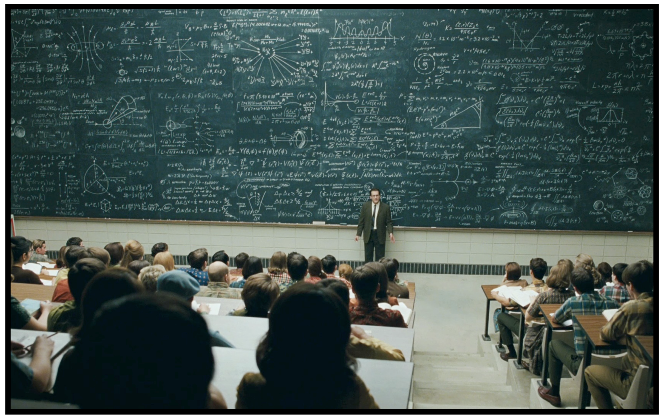
Connections on Facebook

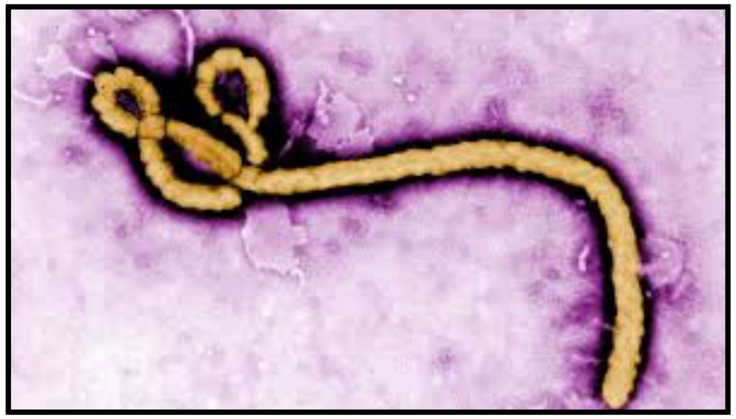

Lausanne transportation map

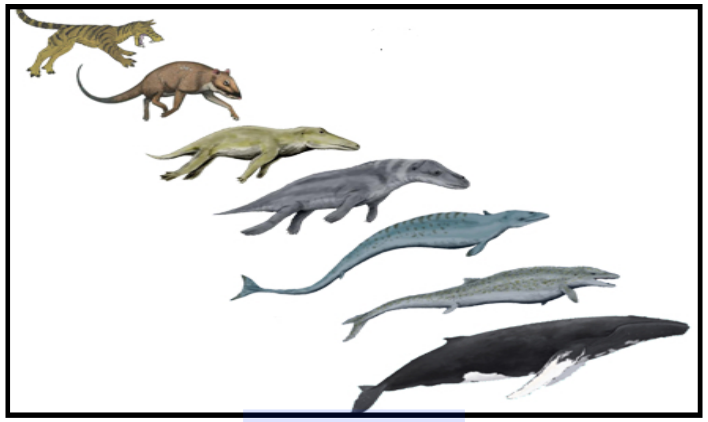


Database of movies


Neurons in the brain






Evolution

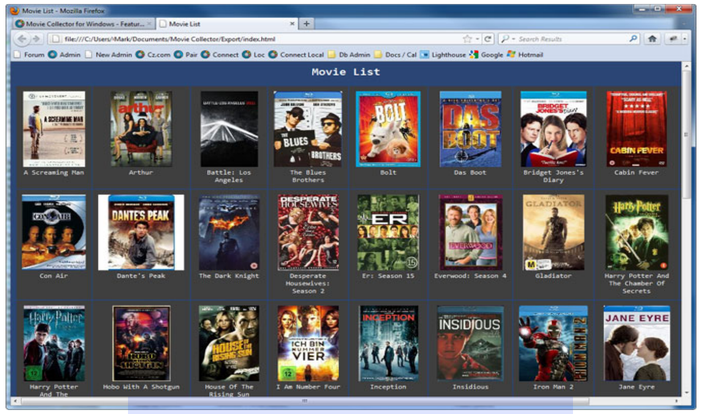

Countries on a map

What do These Examples Have in Common?


People at a dinner party

Connections on Facebook


Lausanne transportation map



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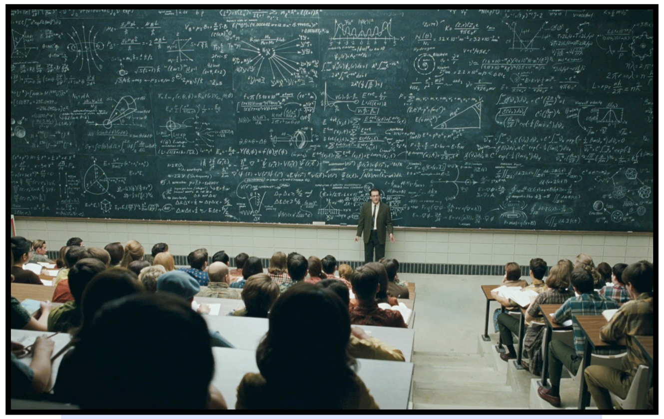
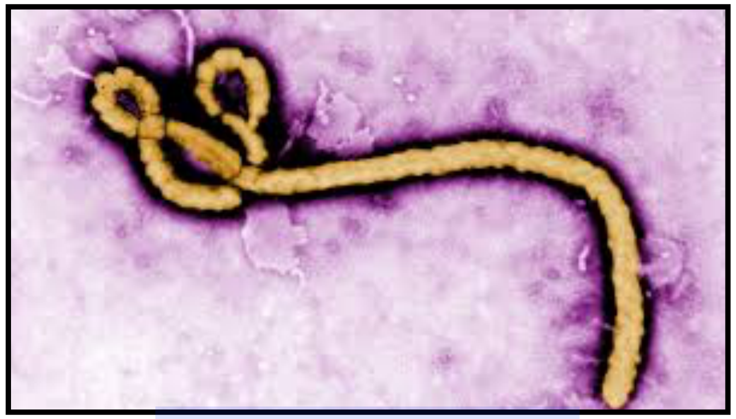
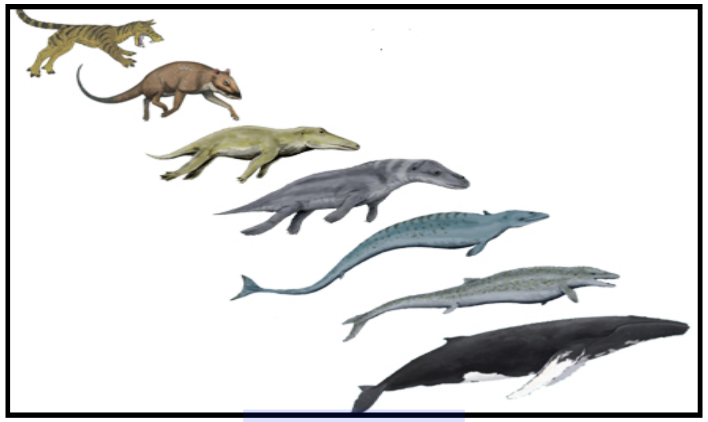

Professors and students


Epidemic outbreak


Evolution


Countries on a map

Common Point

- We have "*items*"
- And we have a "*relationship*" between pairs of items
- Sometimes the relationship is "symmetric"
  - Means that if item1 is in relationship with item2, then item2 is also in relationship with item1
- Sometimes the relationship is "not symmetric"

Dinner Party

Items:
People at a party

Relationship:
If one knows the other

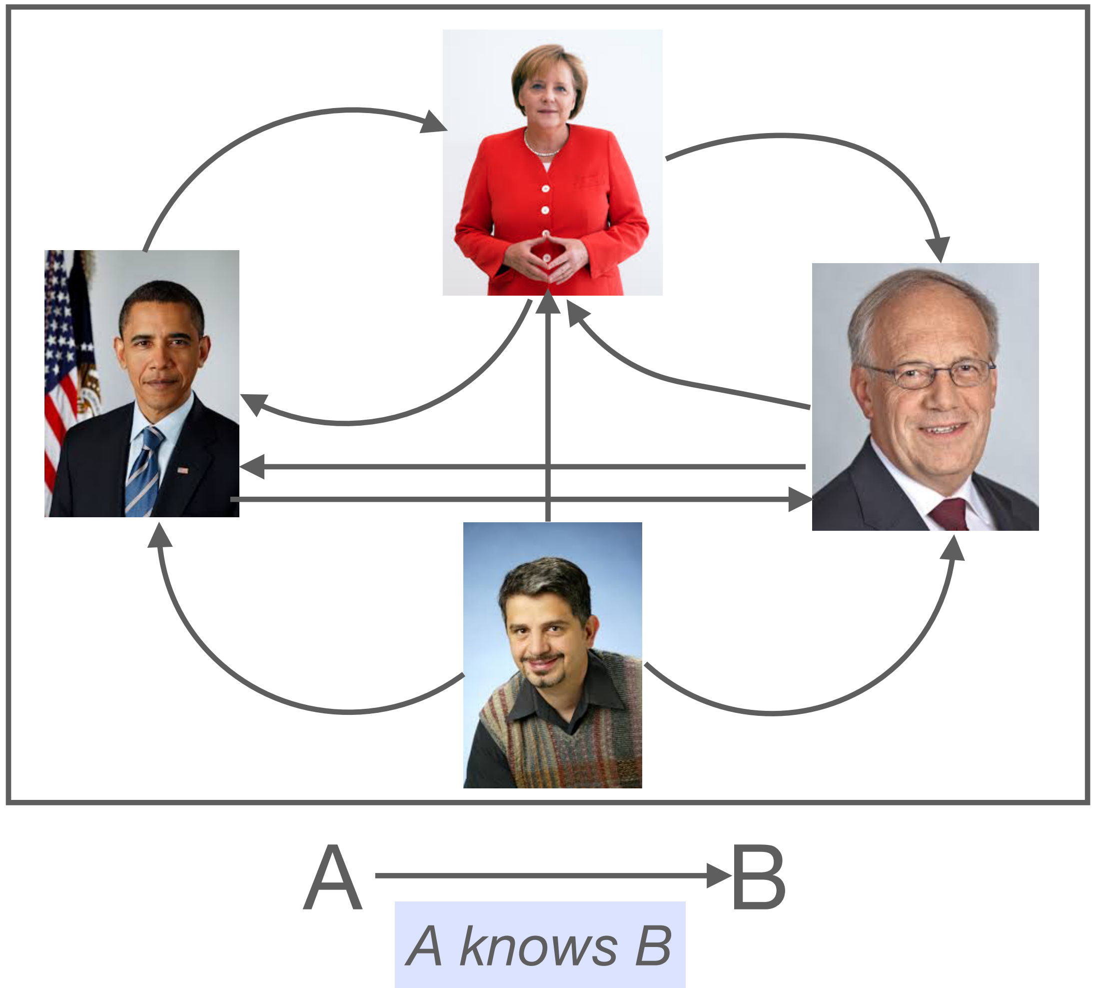
Dinner Party

Items:
People at a party

Relationship:
If one knows the other

Not necessarily symmetric

Dinner Party



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People at a party

Relationship:
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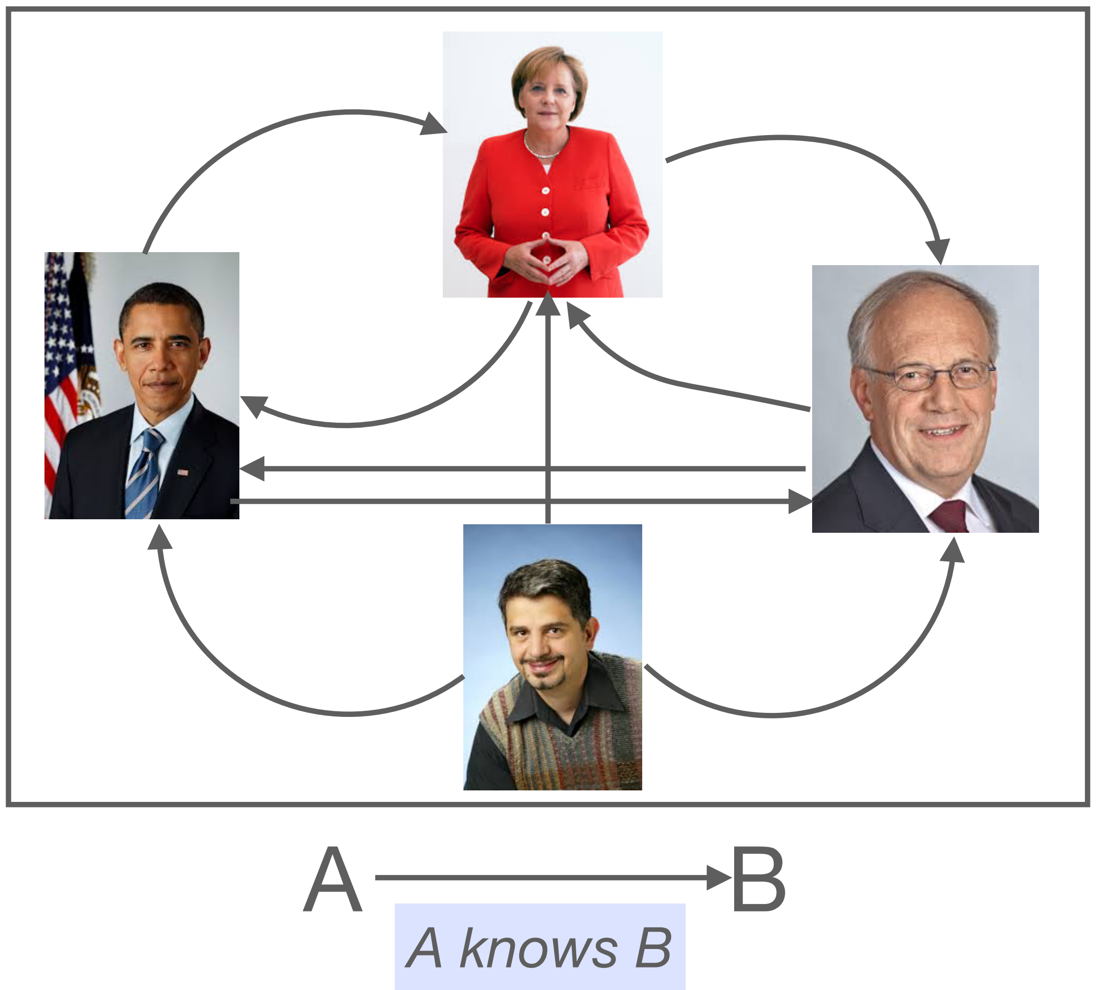
Dinner Party



Items:
People at a party

Relationship:
If one knows the other

Not necessarily symmetric

Facebook

Items:
Facebook users

Relationship:
If one is “friends” with the other

Facebook

Items:
Facebook users

Relationship:
If one is “friends” with the other

Symmetric

Brain

Items:
Neurons

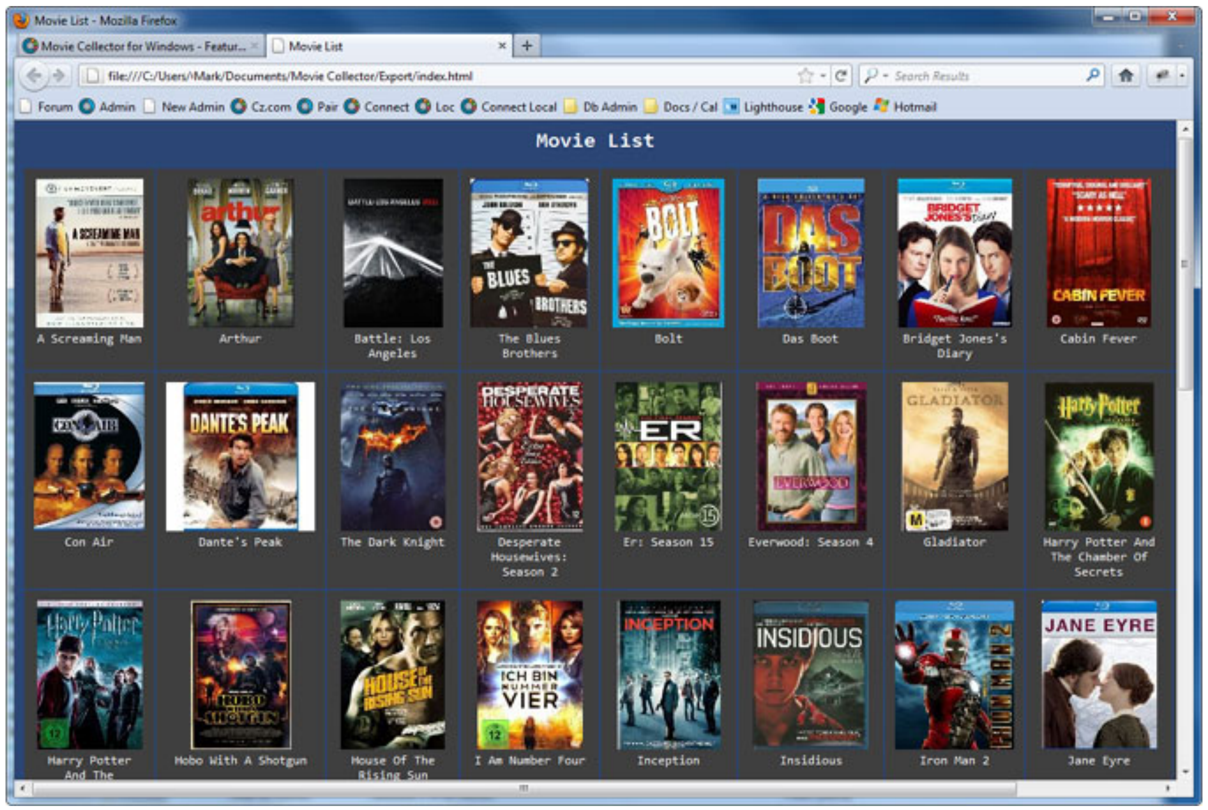
Relationship:
If there is a synaptic connection between them

Brain

Items:
Neurons

Relationship: *Symmetric*
If there is a synaptic connection between them

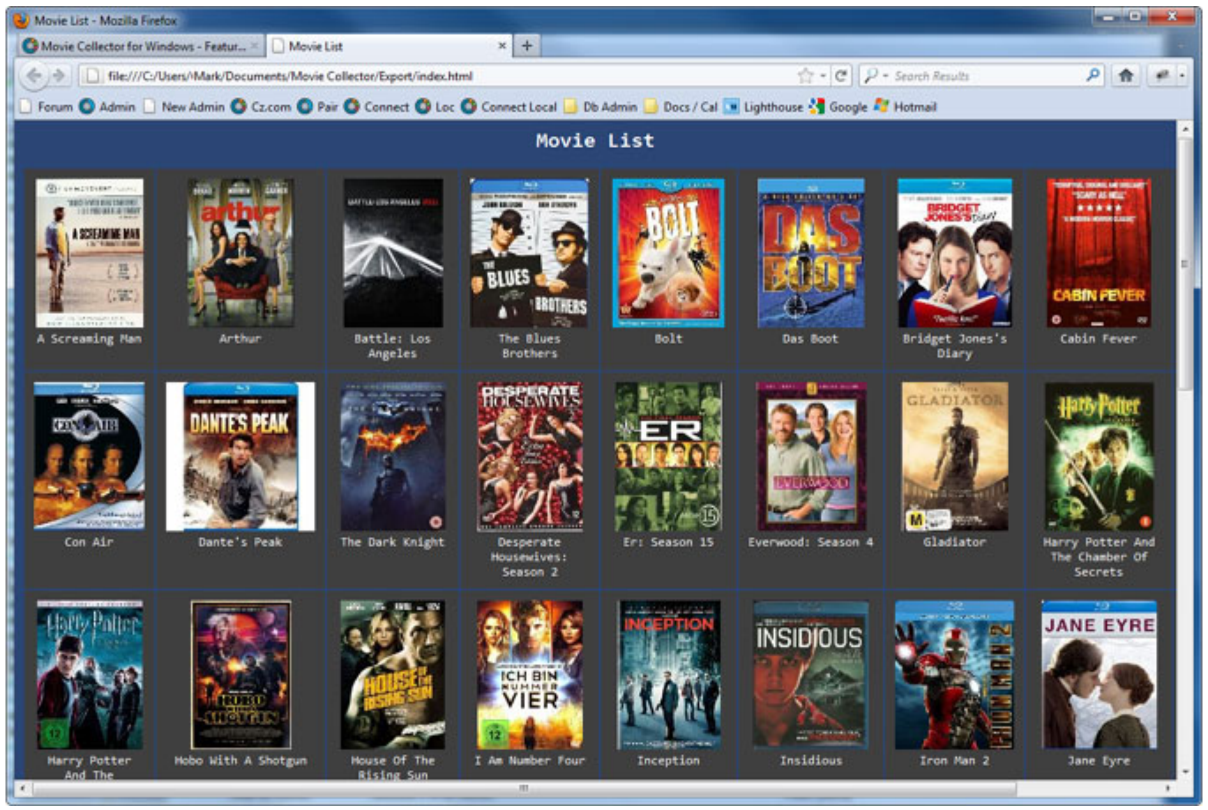
Database of Movies



<u>Items</u>:
Movies

<u>Relationship</u>:
If they belong to the same genre
If they have the same director
If they have an actor in common
……

Database of Movies



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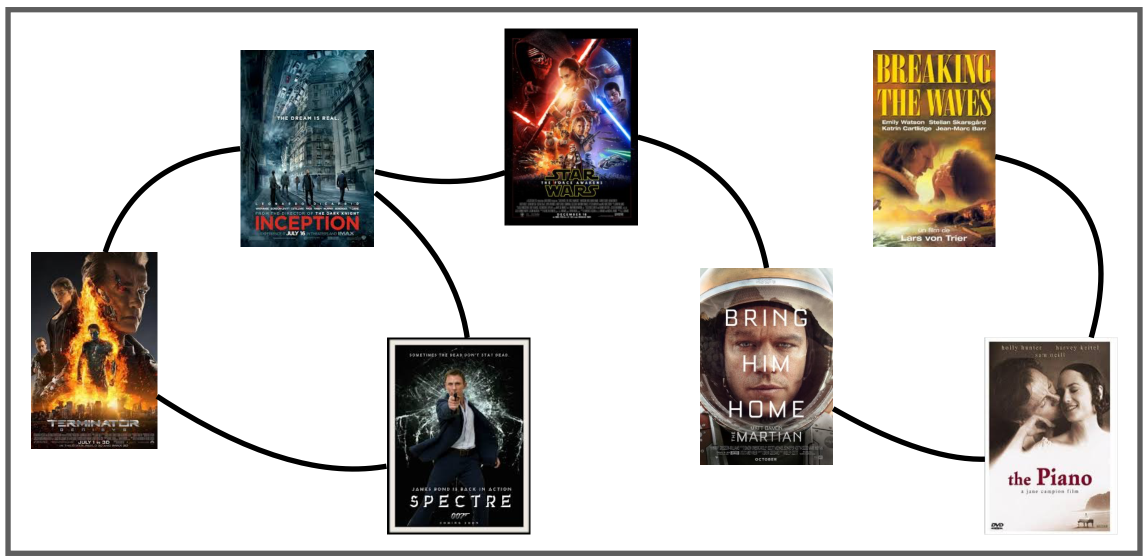
Database of Movies



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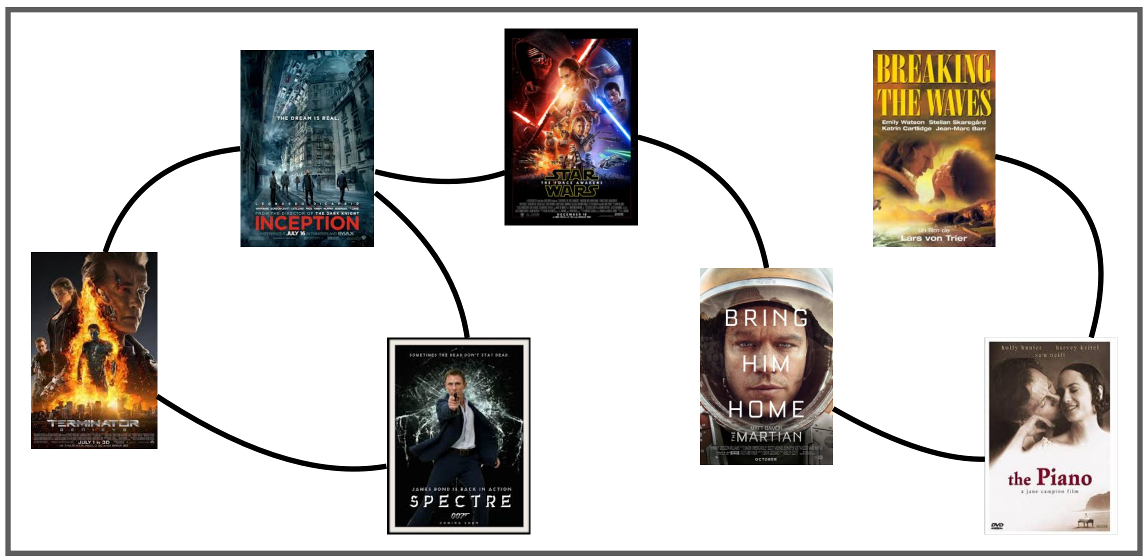

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<u>Relationship</u>:
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Symmetric

Subway

Items:
Metro or bus stations

Relationship:
If there is a metro or a bus line connecting them

Subway

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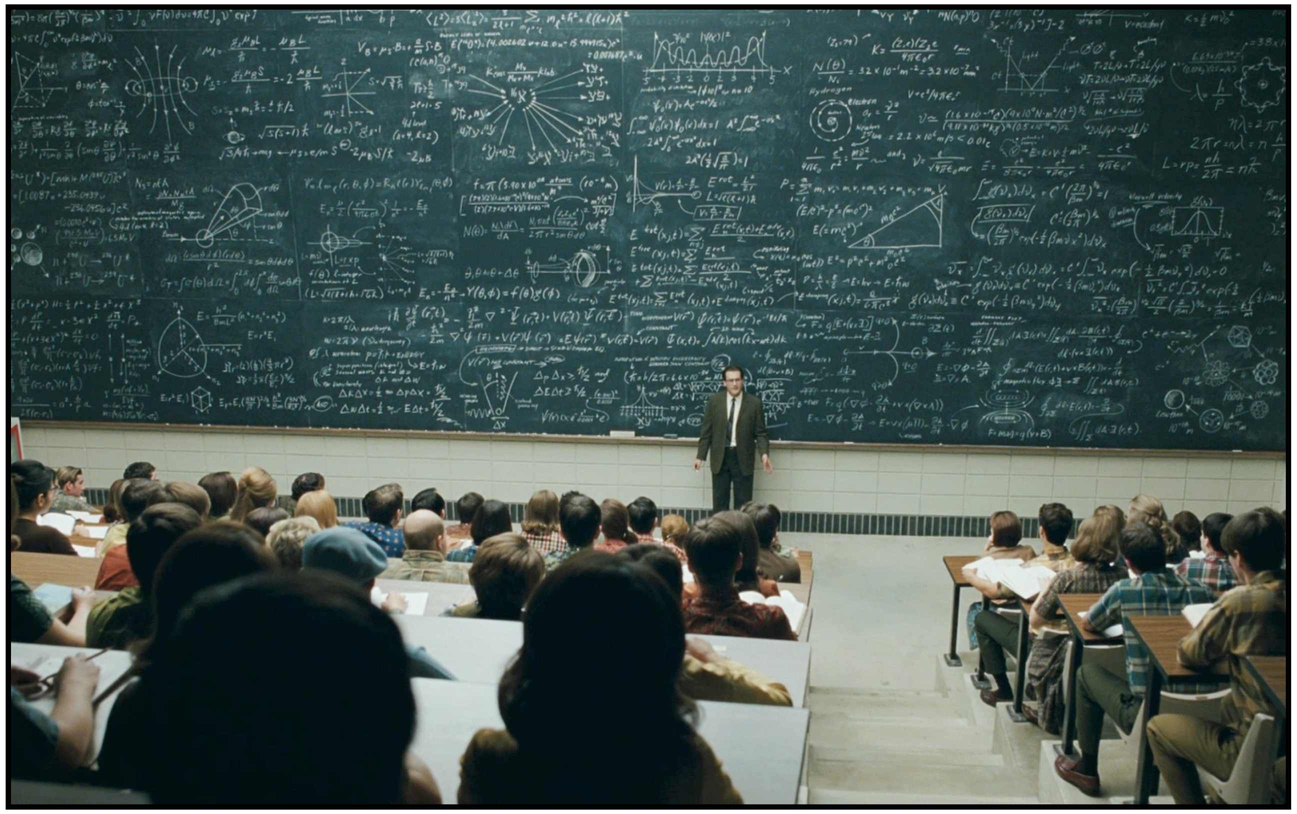
Metro or bus stations

Relationship:
If there is a metro or a bus line connecting them

Symmetric

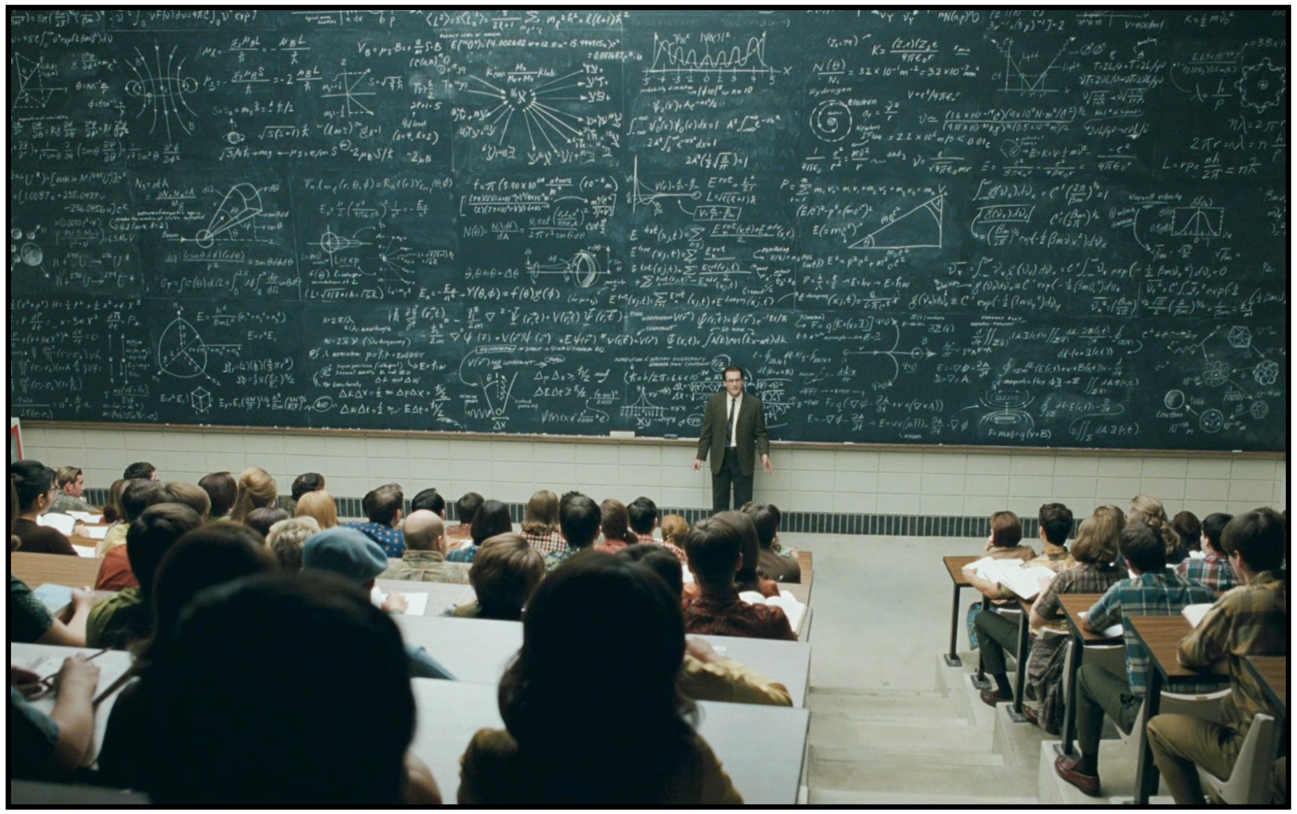
Professors and Students



<u>Items</u>:
Professor/students

<u>Relationship</u>:
If student and professor have been in the same classroom during the semester

Professors and Students



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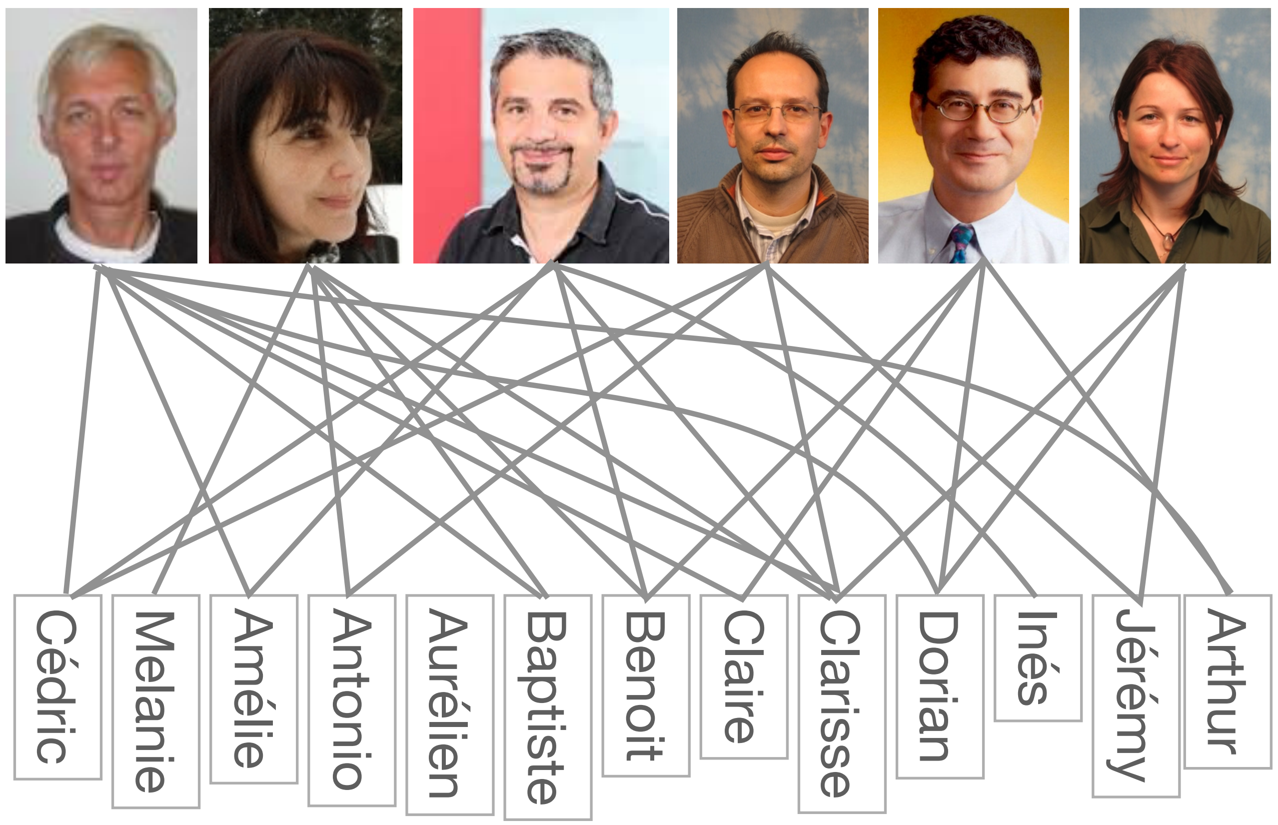

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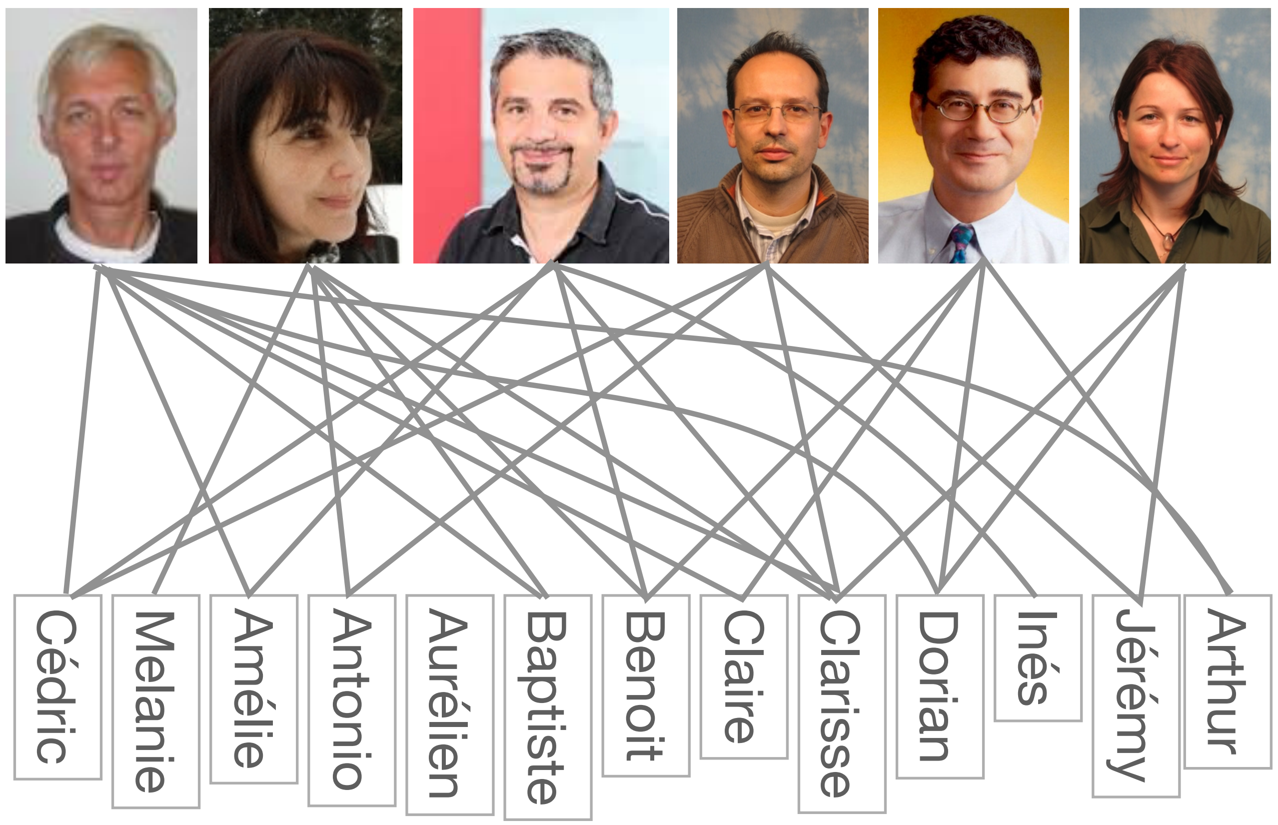
Professors and Students



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Symmetric

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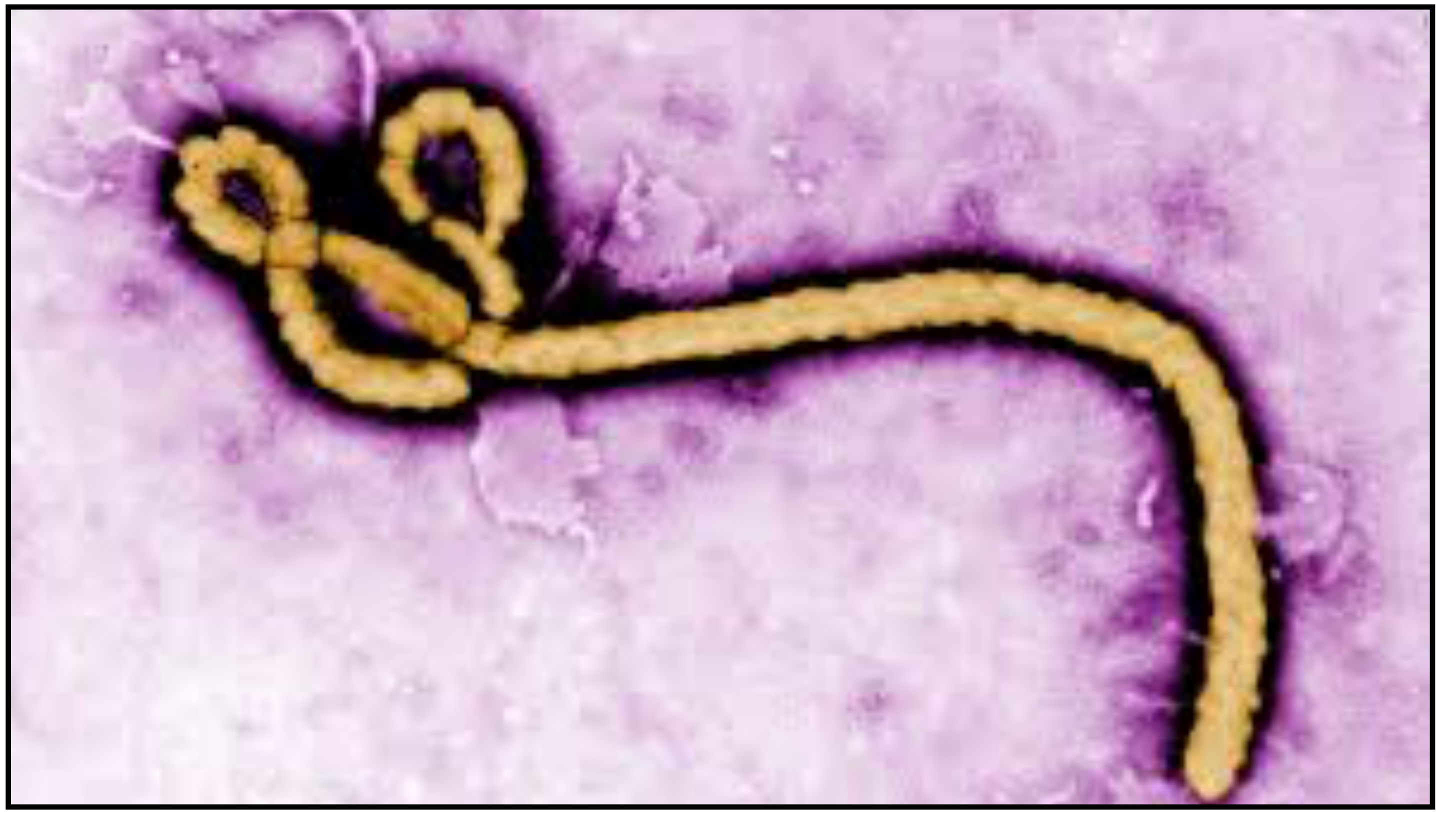
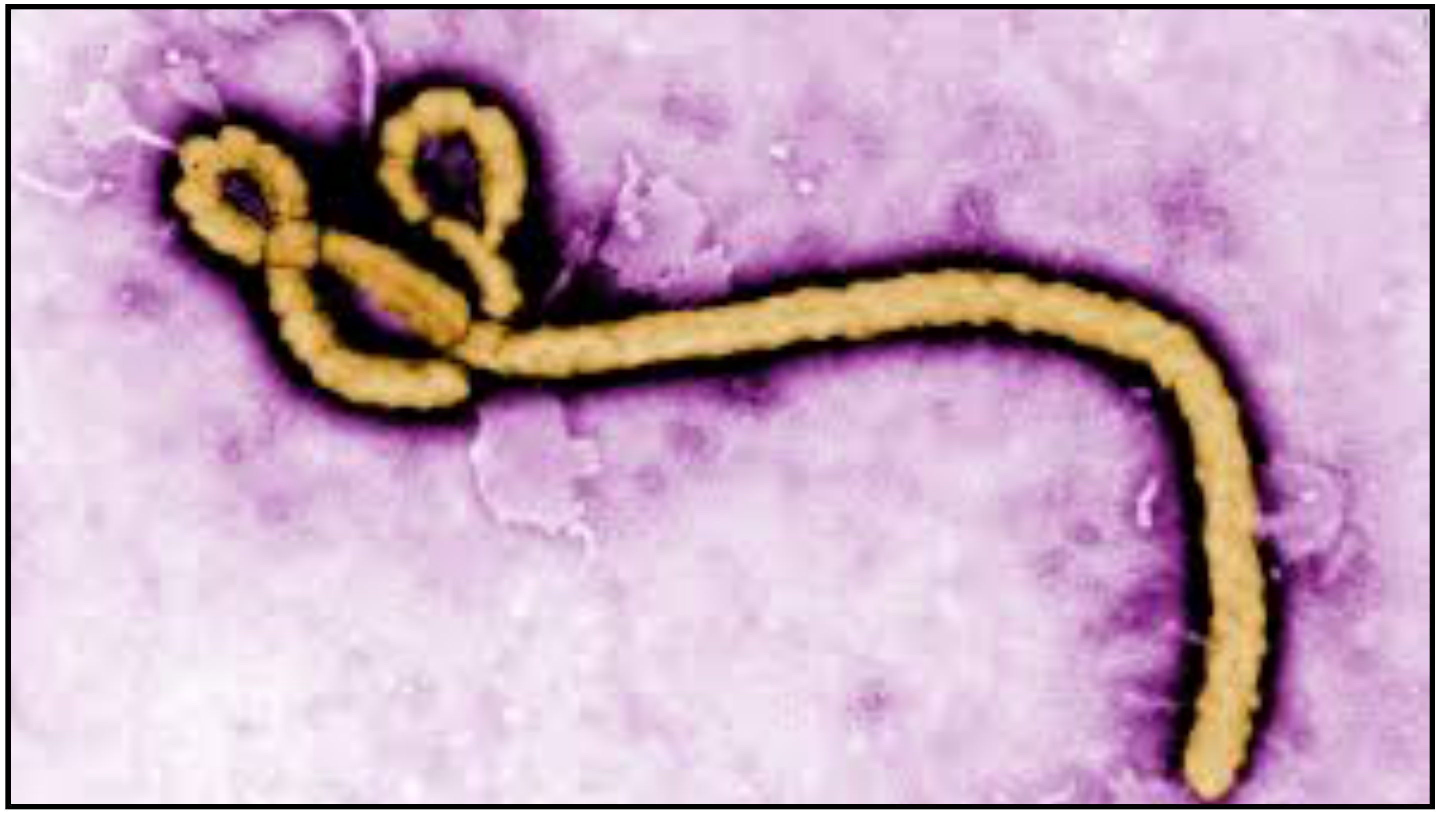
Epidemic Outbreak



Items:
Infected humans

Relationship:
If a person has been infected by another

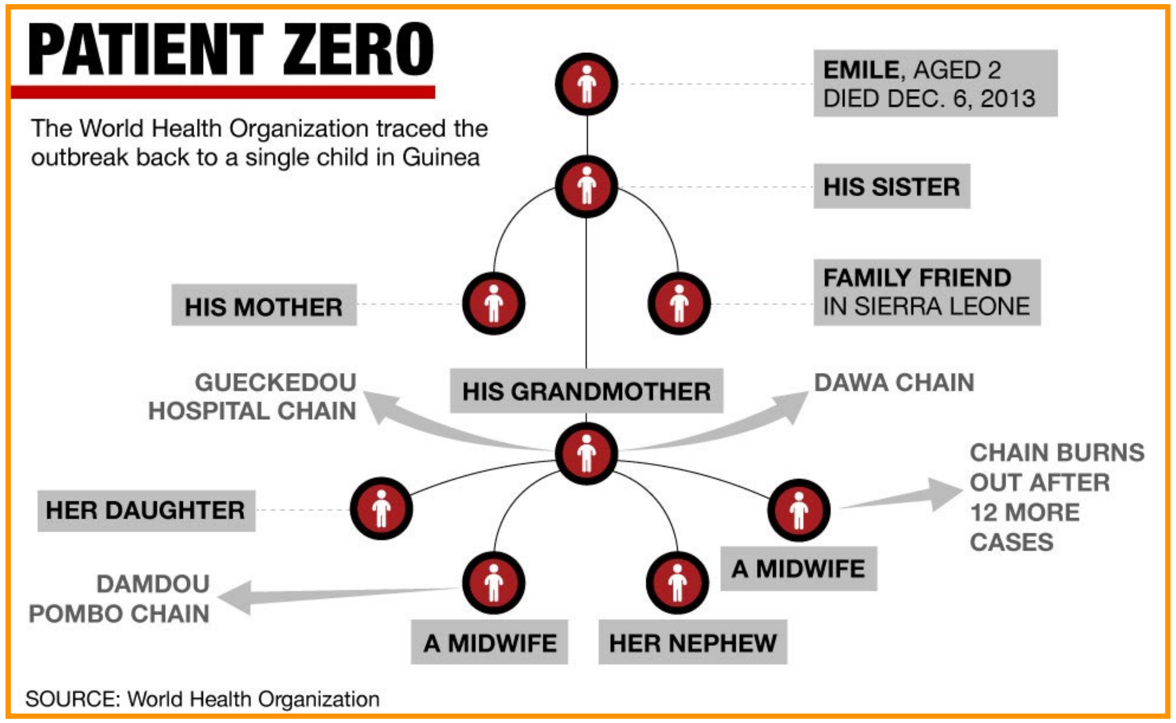
Epidemic Outbreak



Items:
Infected humans

Relationship: *Not symmetric*
If a person has been infected by another

Epidemic Outbreak

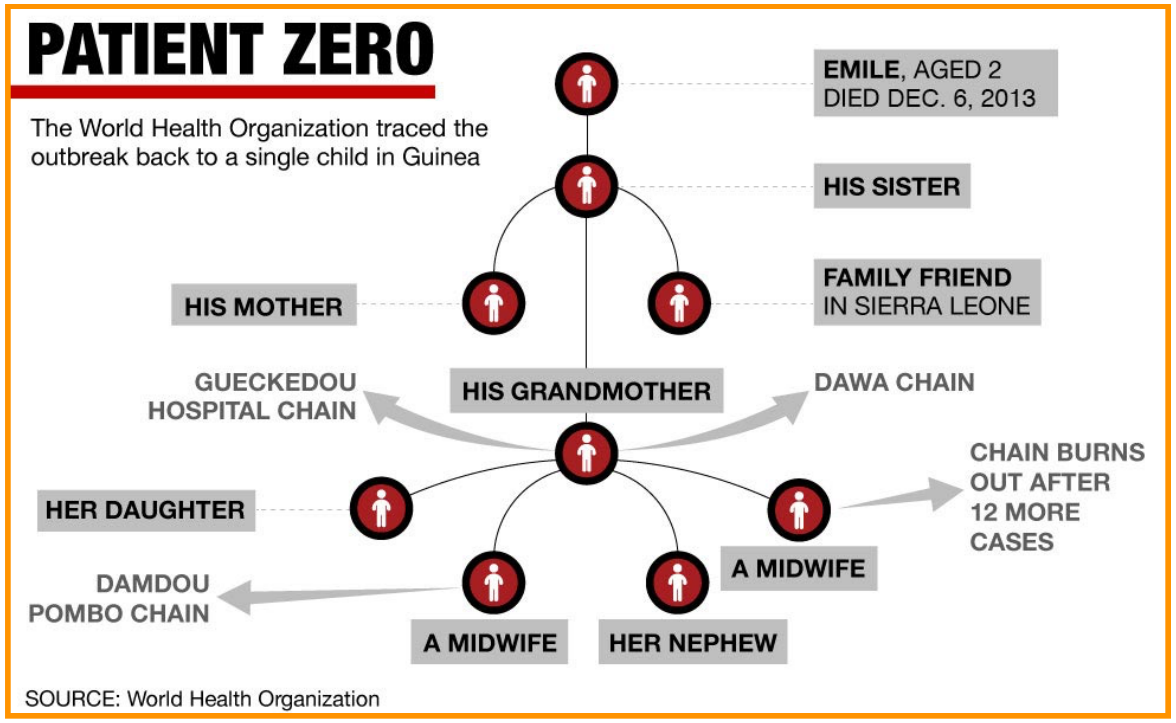


Ebola outbreak 2013

<u>Items</u>:
Infected humans

<u>Relationship</u>:
If a person has been infected by another

Epidemic Outbreak

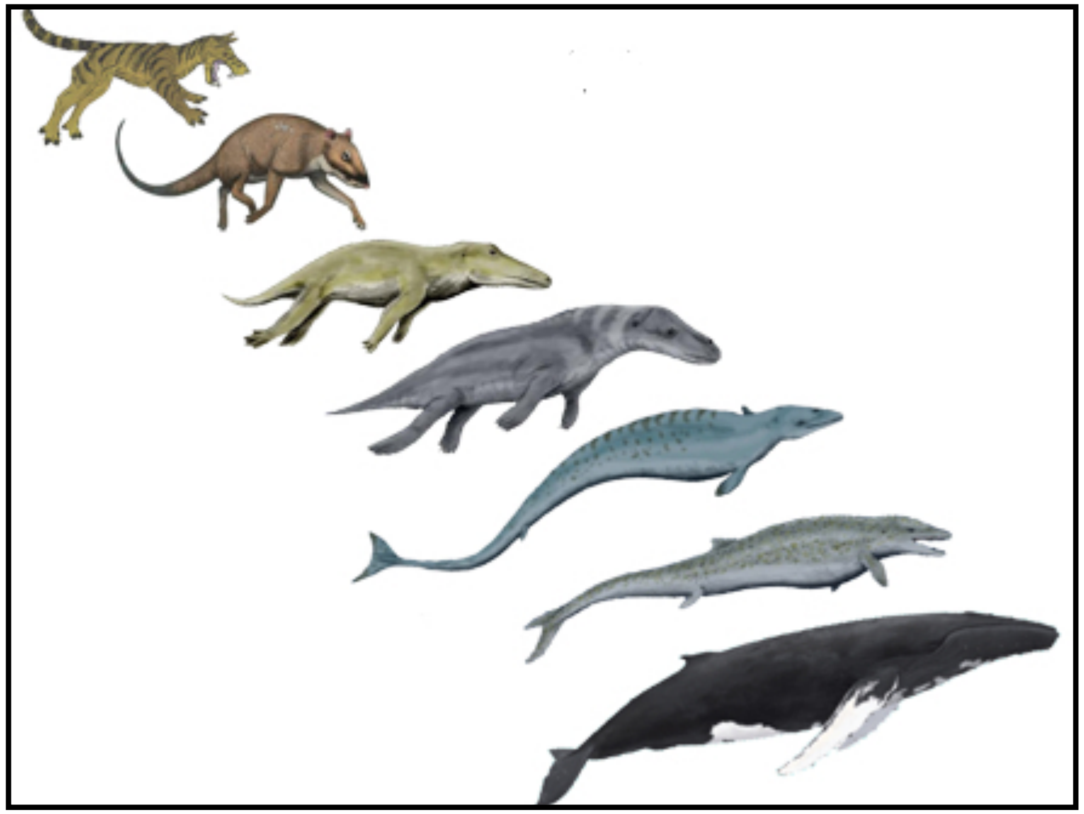


Ebola outbreak 2013

Items:
Infected humans

Relationship: *Not symmetric*
If a person has been infected by another

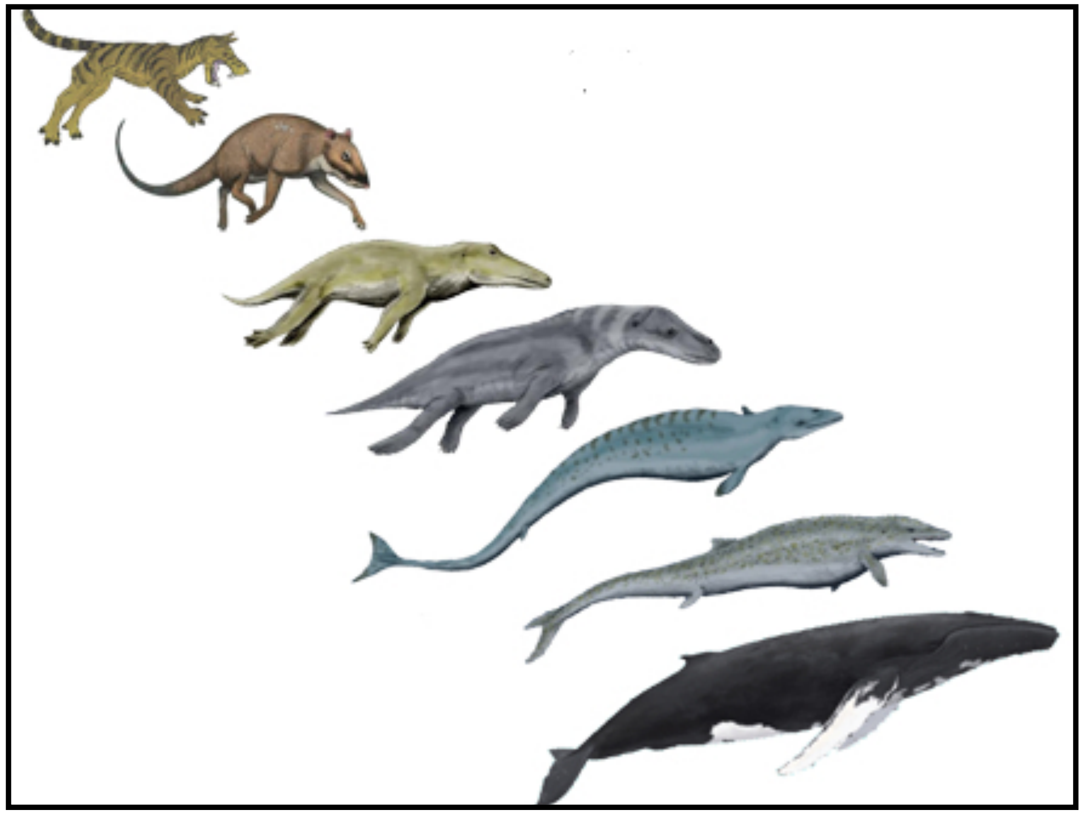
Evolution



<u>Items</u>:
All species that have ever walked the earth

<u>Relationship</u>:
If one species has evolved into another

Evolution



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Relationship:
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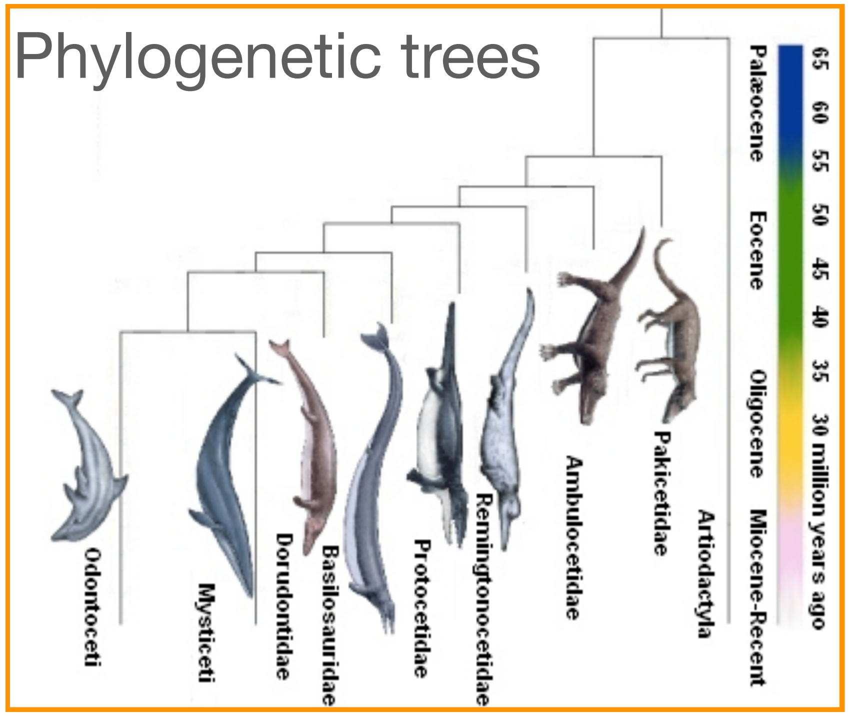
Evolution

Phylogenetic trees



<u>Items</u>:
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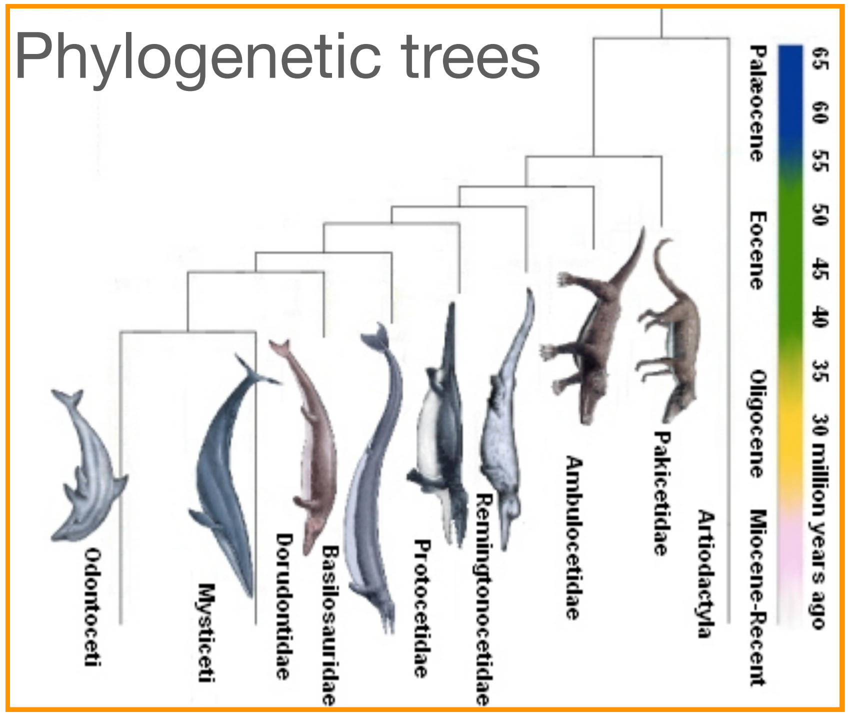
If one species has evolved into another

Evolution



Phylogenetic trees

<u>Items</u>:
All species that have ever walked the earth

<u>Relationship</u>:
If one species has evolved into another

Not symmetric

Map

<u>Items</u>:
Countries

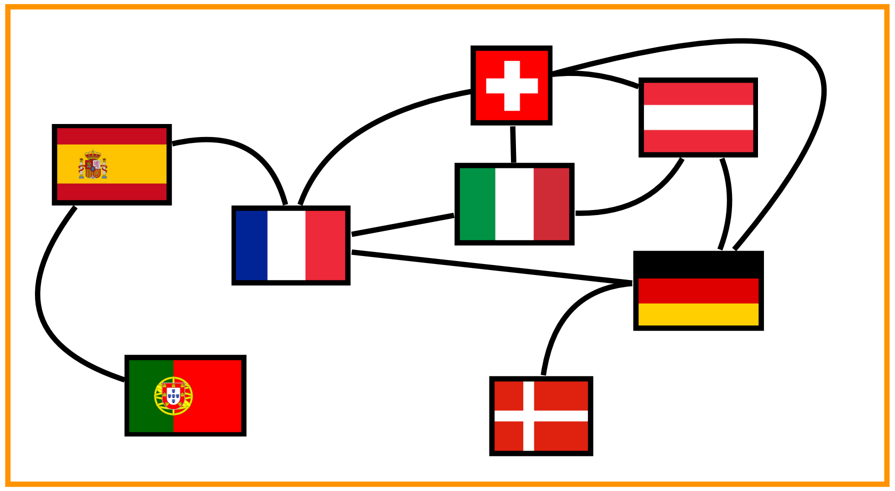
<u>Relationship</u>:
If they have a common border

Map

Items:
Countries

Relationship: *Symmetric*
If they have a common border

Map



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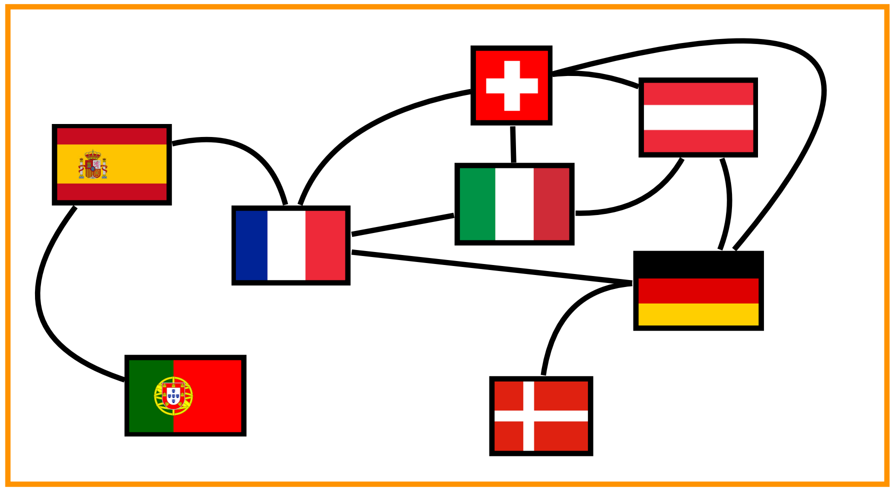


Items:
Countries

Relationship: *Symmetric*
If they have a common border

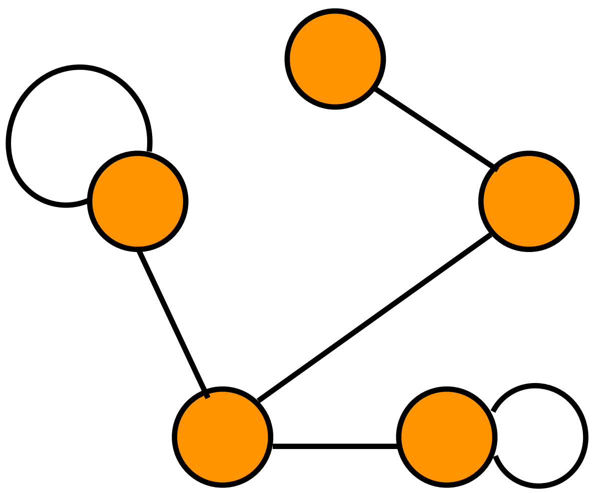
Common Generalization: Graphs

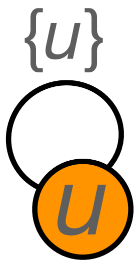
- Need to be able to analyze examples like the above both from a theoretical and a computational point of view
- Abstract out the main concept
- This leads us to graph theory

Undirected Graphs

pp. 617-618

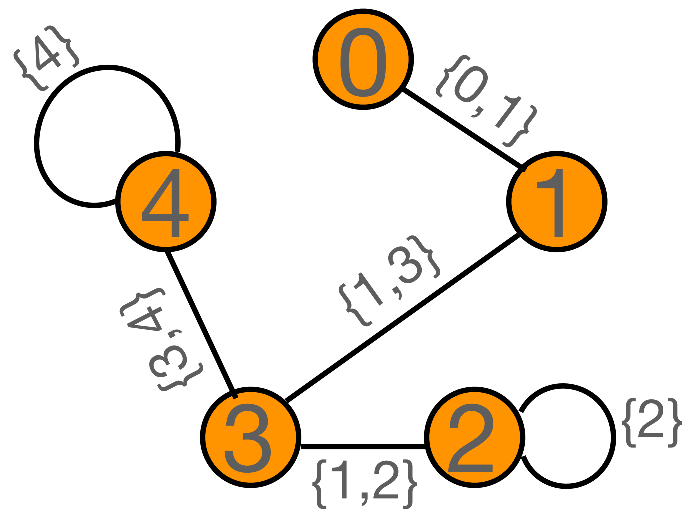
- An *undirected graph* is a set V of *vertices* and a set E of *edges*, $E \subset P(V)$.
- Each element in E has either one or two elements
  - $\{u,v\} \in E$ connects vertices $u, v \in V$.
    - ➡ u,v are called *endpoints* of the edge, and called *adjacent* or *neighbors*
  - $\{u\} \in E$ is a *loop*. Graph with no loops is called *simple*.



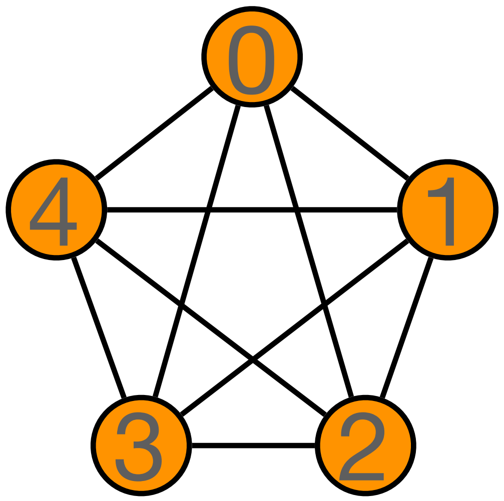


Examples of Undirected Graphs

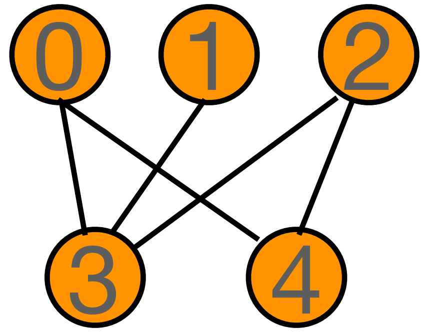
- $V = \{0,1,2,3,4\}$
- $E = \{\{0,1\},\{2\},\{1,3\},\{3,4\},\{2,3\},\{4\}\}$



- $V = \{0,1,2,3,4\}$
- $E = \{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{1,2\},\{1,3\},$ $\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$



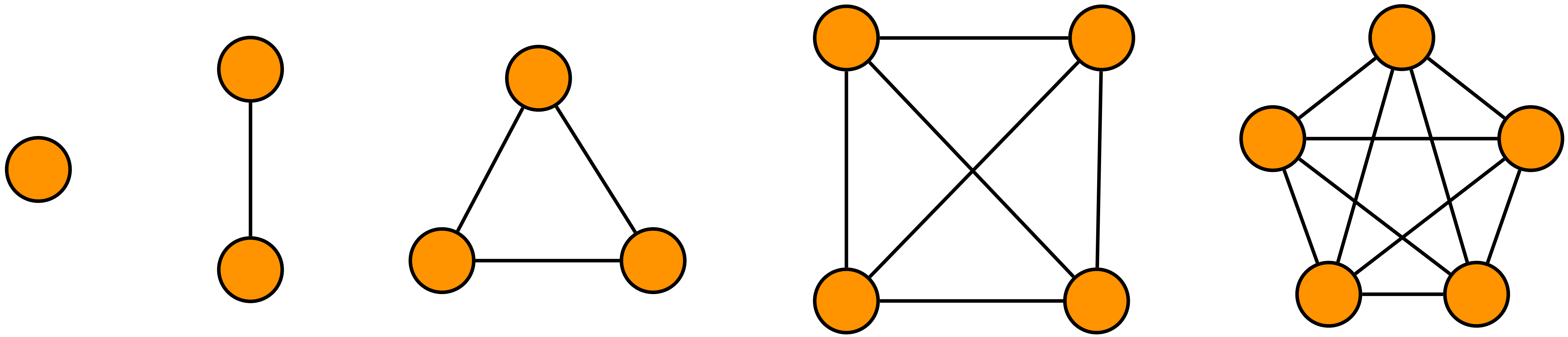
- $V = \{0,1,2\} \cup \{3,4\}$
- $E = \{\{0,3\},\{0,4\},\{1,3\},\{2,4\},\{2,3\}\}$



Complete Graphs

pp. 630

- Every pair of nodes is connected.
- Complete graph on n nodes is denoted by K_n.
- Formally: $K_n = (V,E)$, where
  - $|V|=n$ and
  - $E = \{S|\ S \subseteq V \text{ and } |S| = 2\}$.



K_1 K_2 K_3 K_4 K_5

Bipartite Graphs

pp. 633-635

- Two different classes of nodes
- Edges only between nodes in different classes
- Formally: $B = (V_1 \cup V_2, E)$, $E \subseteq V_1 \cup V_2$, $E \not\subset V_1$ and $E \not\subset V_2$, for all $e \in E$: $|e| = 2$

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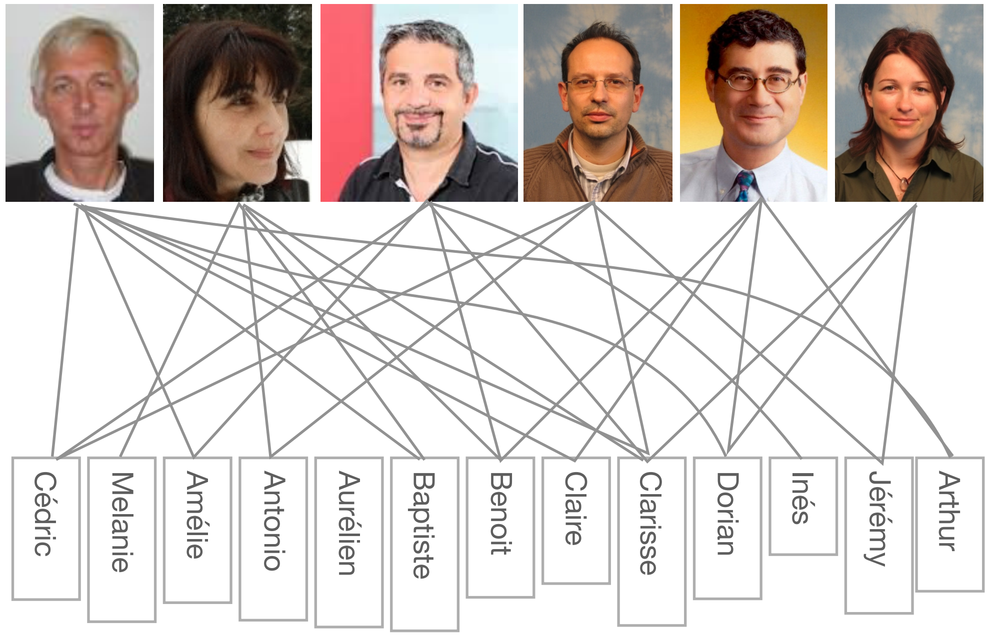

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pp. 633-635

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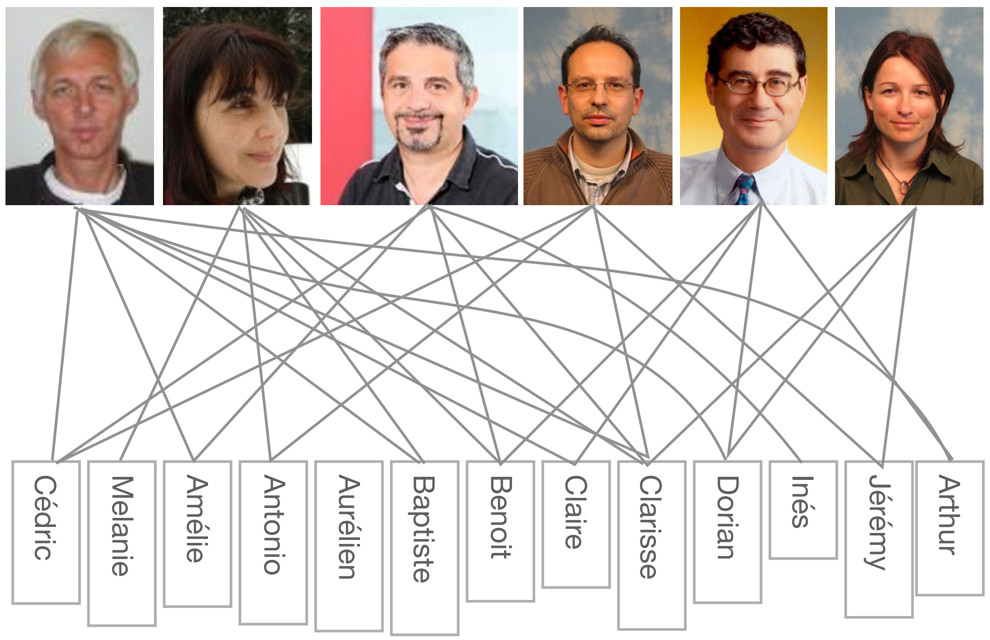
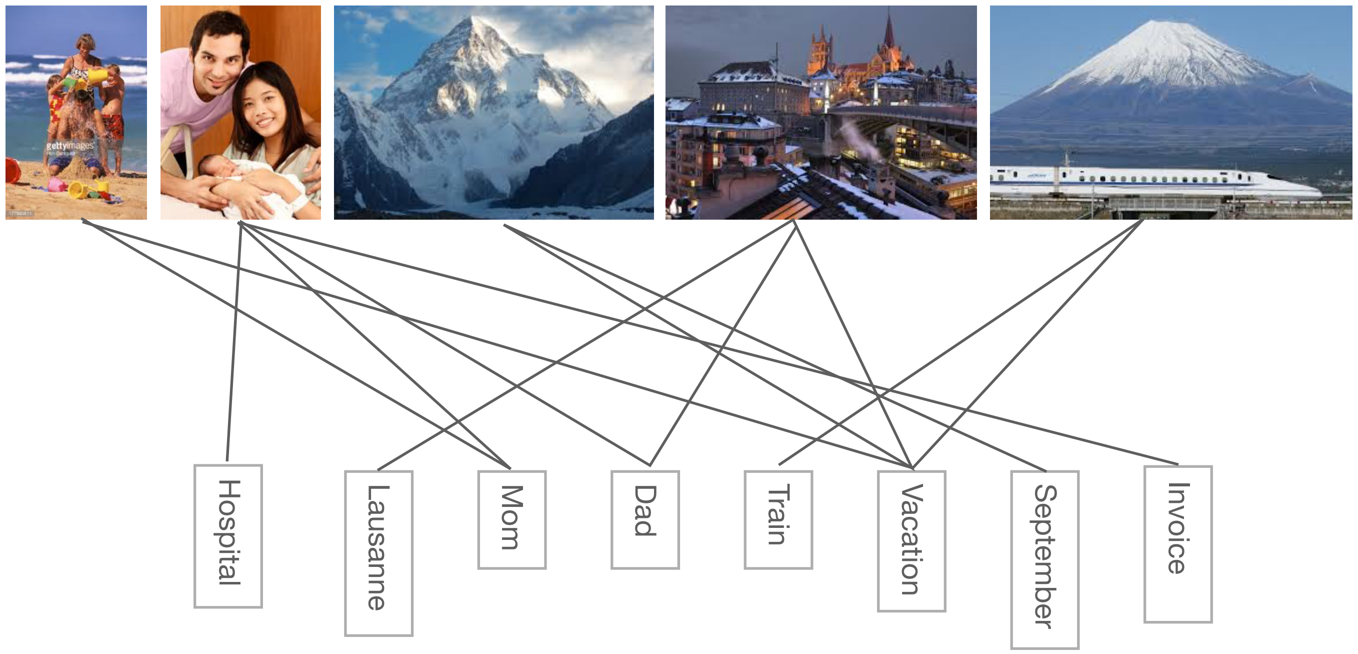


Keywords and files



Subgraphs

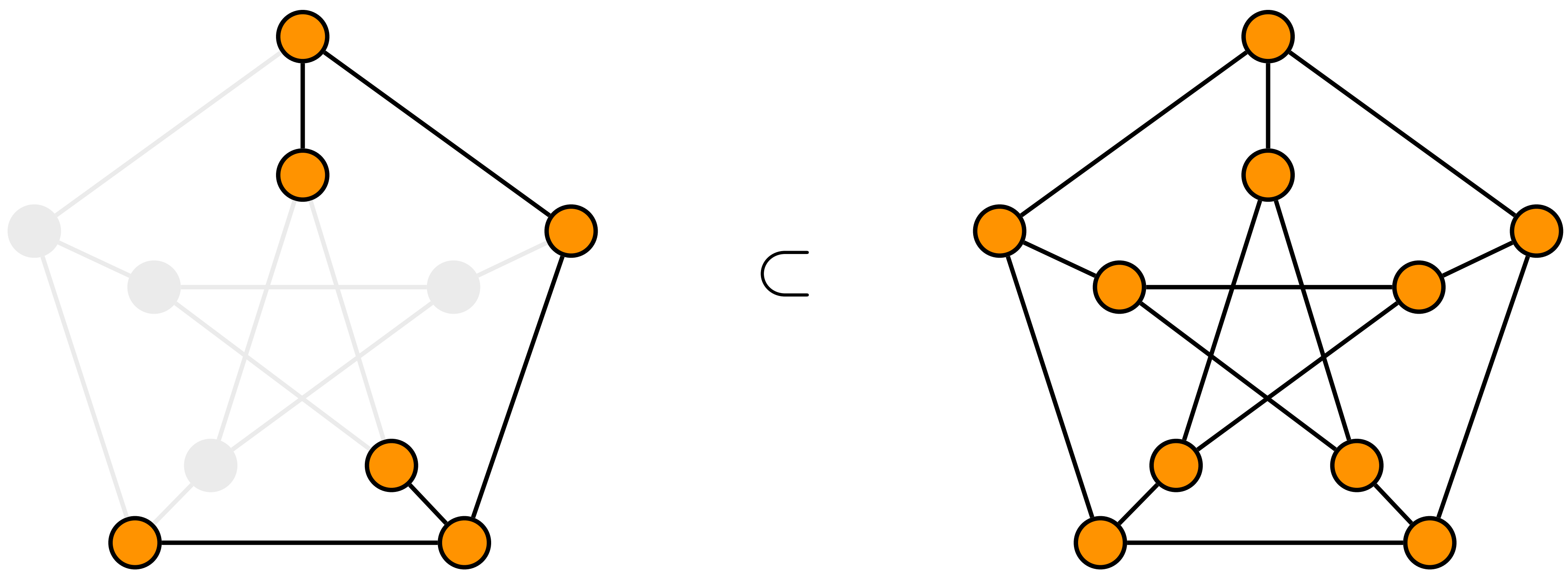
pp. 639

- A subgraph of a graph (V,E) is a graph (V',E') where
  - V' is a subset of V
  - E' is a subset of E

Subgraphs

pp. 639

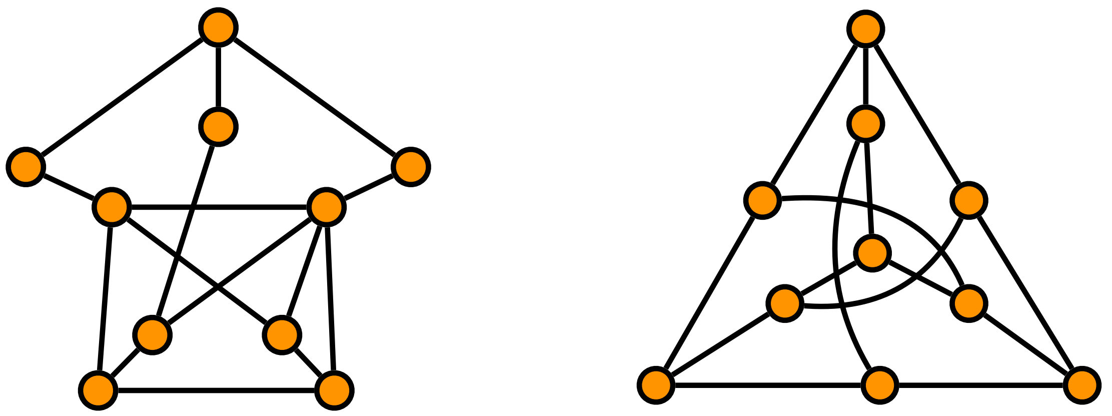
- A subgraph of a graph (*V*,*E*) is a graph (*V'*,*E'*) where
  - *V'* is a subset of *V*
  - *E'* is a subset of *E*



Isomorphism

pp. 647-650

- Two graphs are isomorphic if one can be transformed into another through renaming of nodes
- Formally:
  - (V,E) is isomorphic to (V',E') if there is a bijection $f: V \leftrightarrow V'$ such that $\{u,v\} \in E$ iff $\{f(u),f(v)\} \in E$



Isomorphic?

Isomorphism

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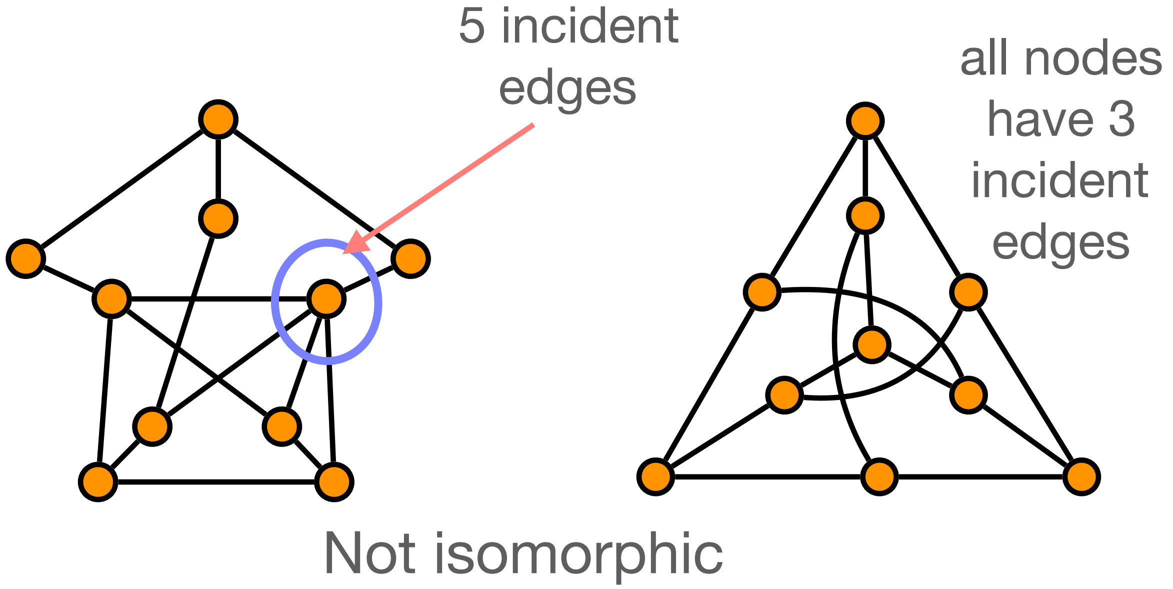

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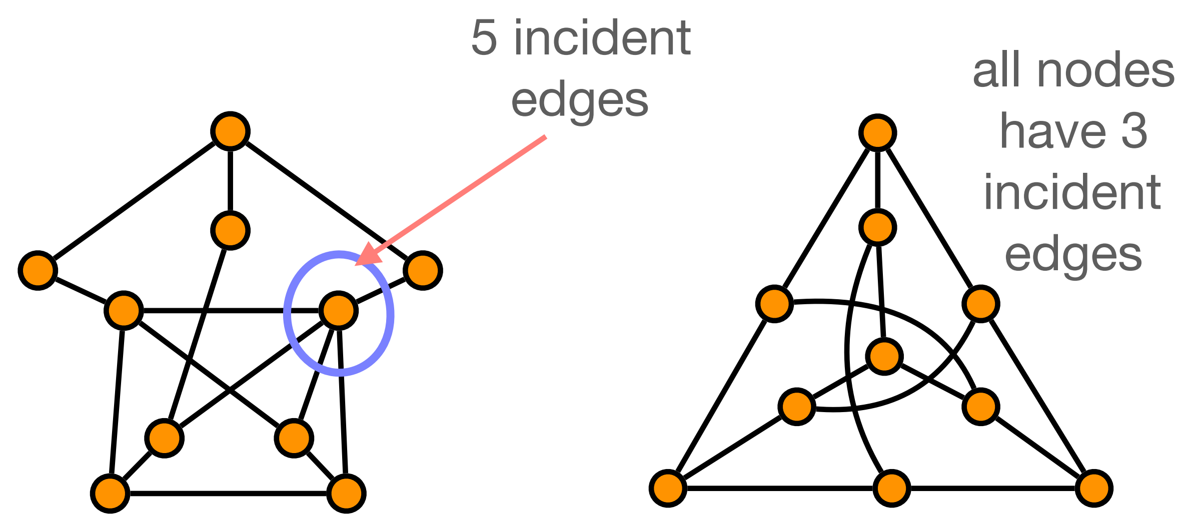
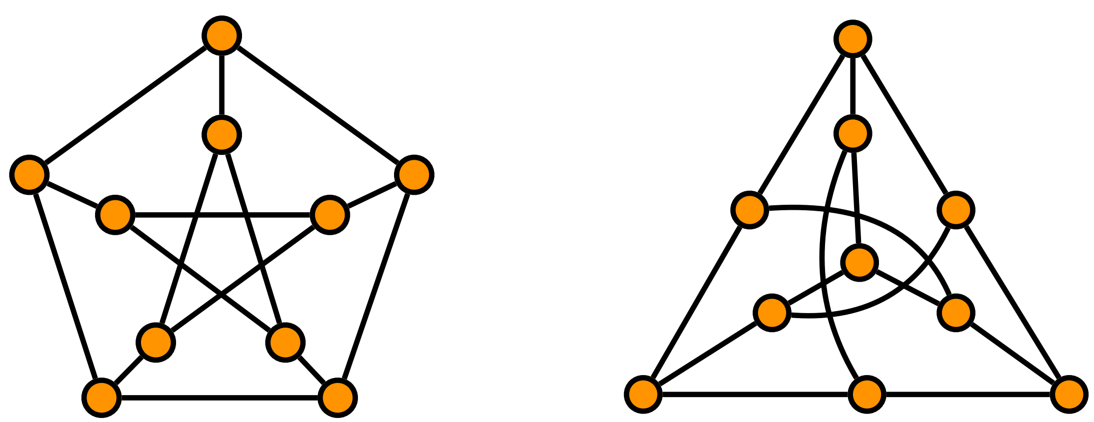


Not isomorphic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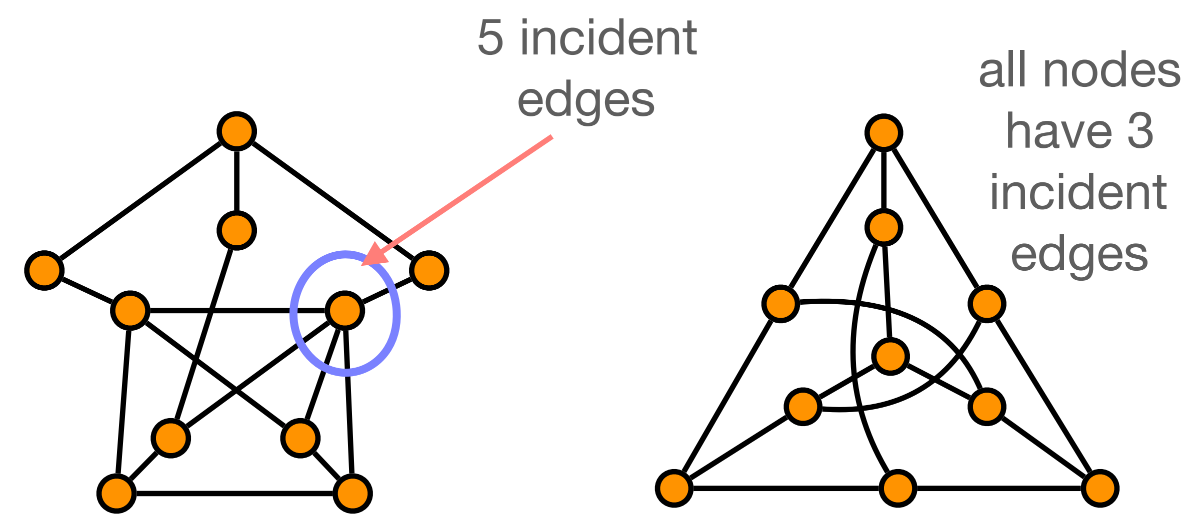

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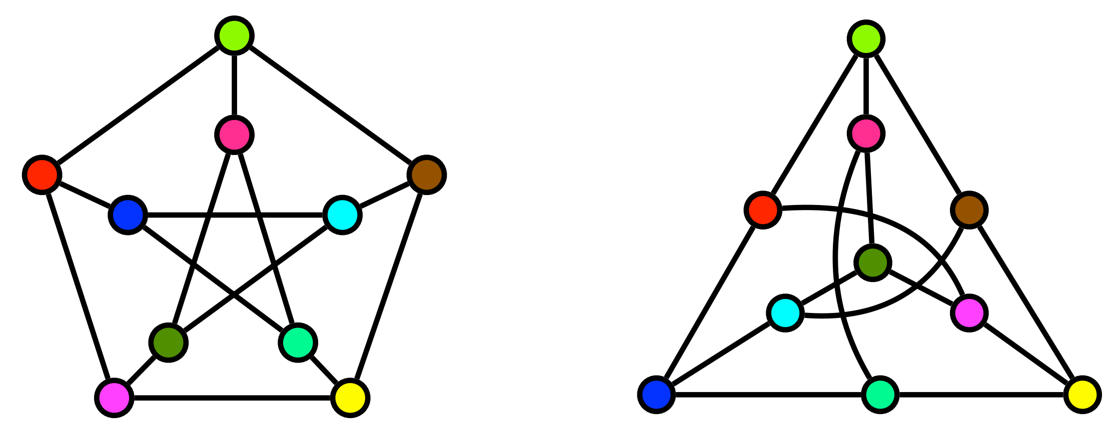
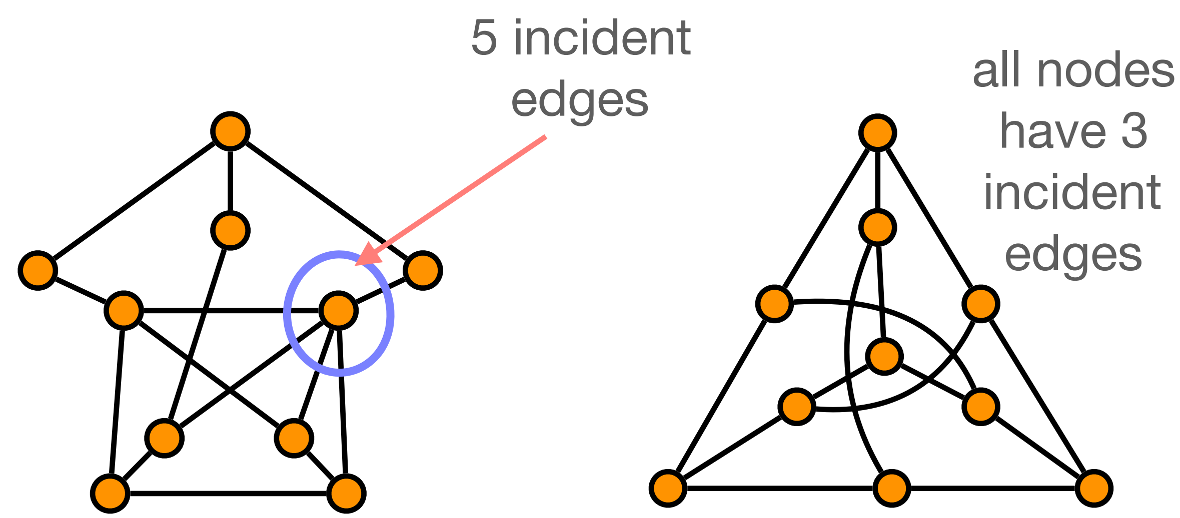


Isomorphic?
Yes

Isomorphism

pp. 647-650

- Two graphs are isomorphic if one can be transformed into another through renaming of nodes
- Formally:
  - (V,E) is isomorphic to (V',E') if there is a bijection $f: V \leftrightarrow V'$ such that $\{u,v\} \in E$ iff $\{f(u),f(v)\} \in E'$
- In general it is difficult to decide isomorphism computationally
  - Though there has been a very recent breakthrough by L. Babai



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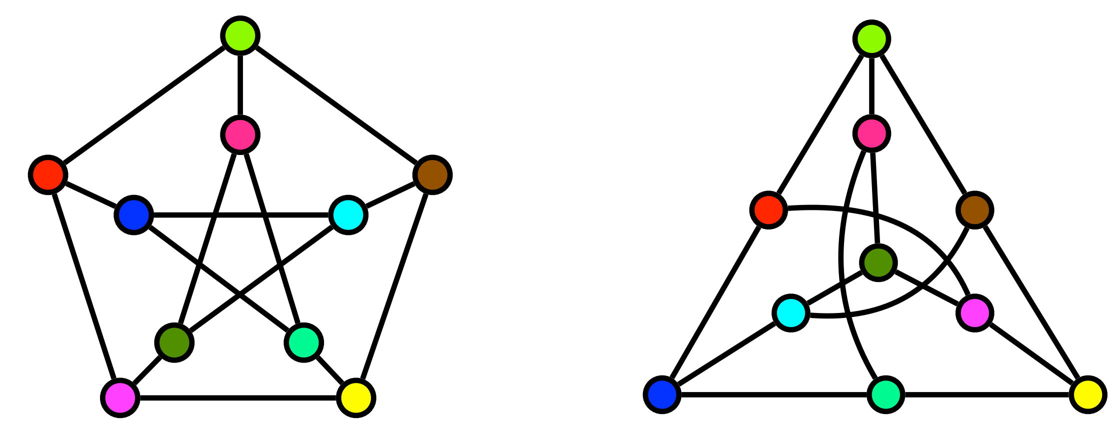
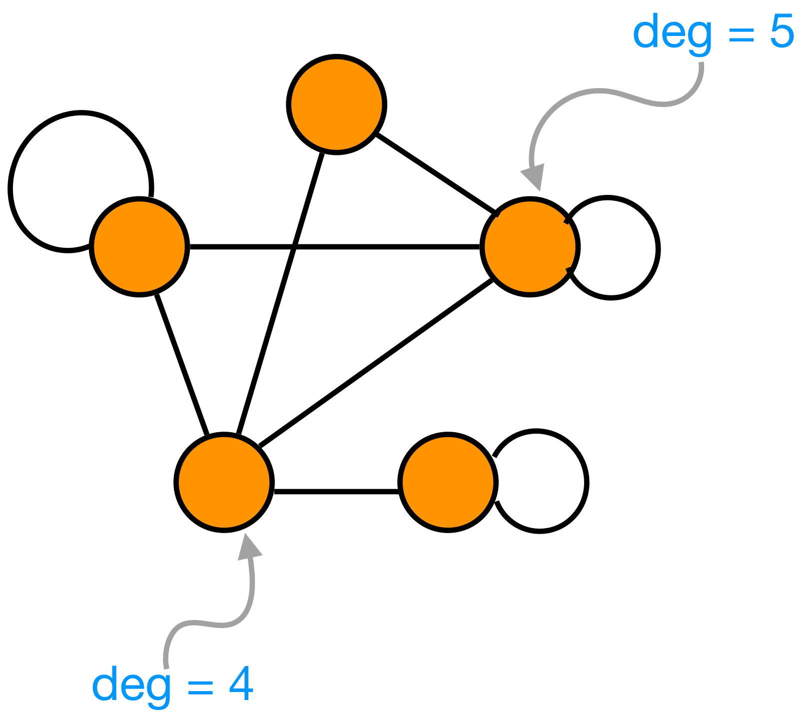


Isomorphic?
Yes

Degree

pp. 627

- The *degree* of a node is the number of non-loop edges incident to the node + 2*number of incident loops
- Formally: $\deg(u) = |\{\{u,v\} \in E\}| + 2|\{\{u\} \in E\}|$



Simple Degree Formulas

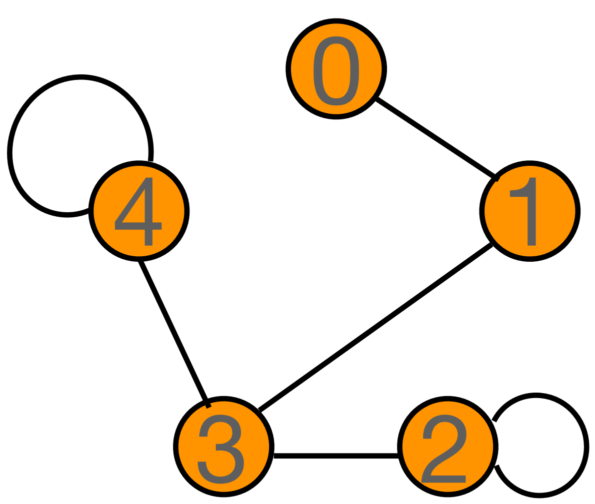
pp. 629

- $G = (V,E)$ graph. Sum of degrees = $2|E|$.
  - Summing degrees counts each edge twice (even the loops).
- $G = (V,E)$ graph. Number of vertices of odd degree is even.
  - $2|E|$ = sum of degrees of even-degree vertices + sum of degrees of odd-degree vertices.
  - First sum is even (sum of even numbers), and final result is even ($2|E|$), so sum of degrees of odd-degree vertices is even.
  - If there were an odd number of odd-degree vertices, then the sum of their degrees would be odd.

Adjacency Matrix

pp. 644-646

- G = (*V*,*E*) graph, $V=\{v_1,\ldots,v_n\}$. An *adjacency matrix* for *G* is an *n* x *n-matrix* $A=(a_{ij})$ such that
  - For $i \neq j$ $a_{ij} = 1$ if $\{v_i, v_j\} \in E$, and $a_{ij} = 0$ otherwise.
  - For $i=j$ $a_{ii} = 2$ if $\{v_i\} \in E$.
- Note that the adjacency matrix depends on the ordering of the elements of *V* (hence is not unique).



	0	1	2	3	4
0	0	1	0	0	0
1	1	0	0	1	0
2	0	0	2	1	0
3	0	1	1	0	1
4	0	0	0	1	2

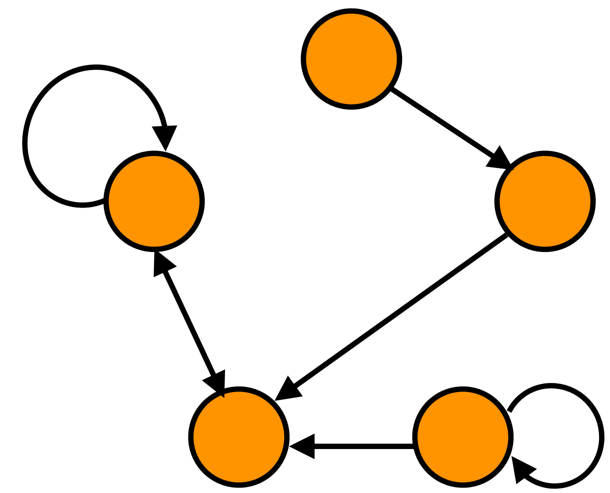
Sum of entries in row *i* is the degree of node v_i

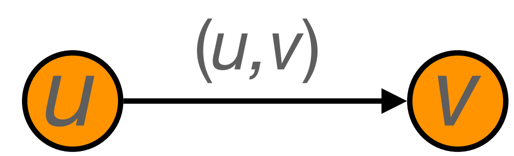
Matrix is always symmetric

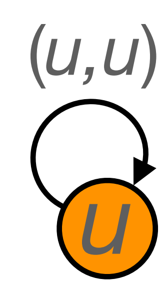
Directed Graphs

pp. 619

- A *directed graph* is a set V of *vertices* and a set E of *edges*, $E \subset V \times V$.
  - $(u,v) \in E$ connects vertices $u, v \in V$.
  - u is the starting point and v the *endpoint* of the edge
- A directed graph on a set V is also called a relation on V.



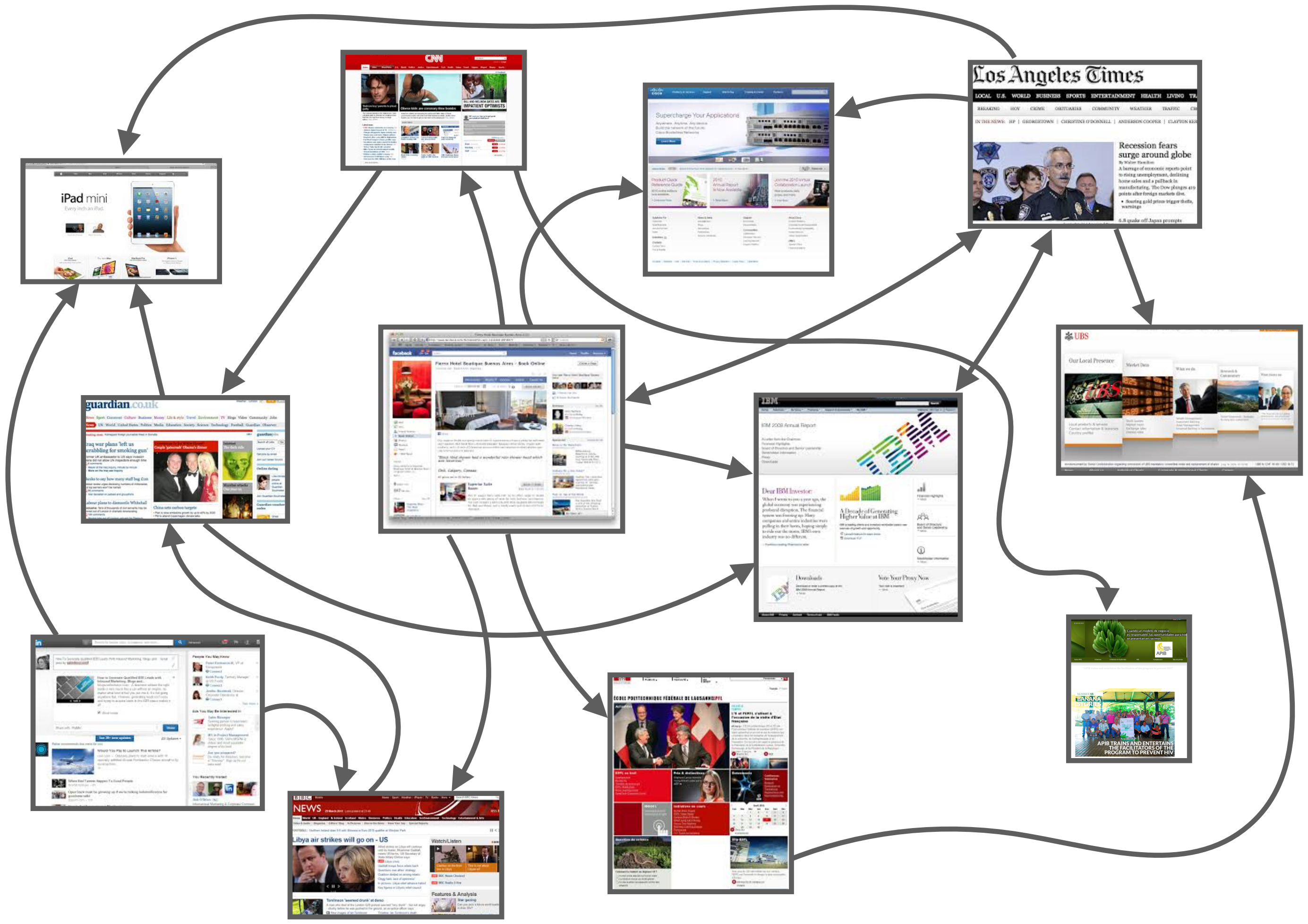




Example of a Directed Graph

Vertices:
Websites

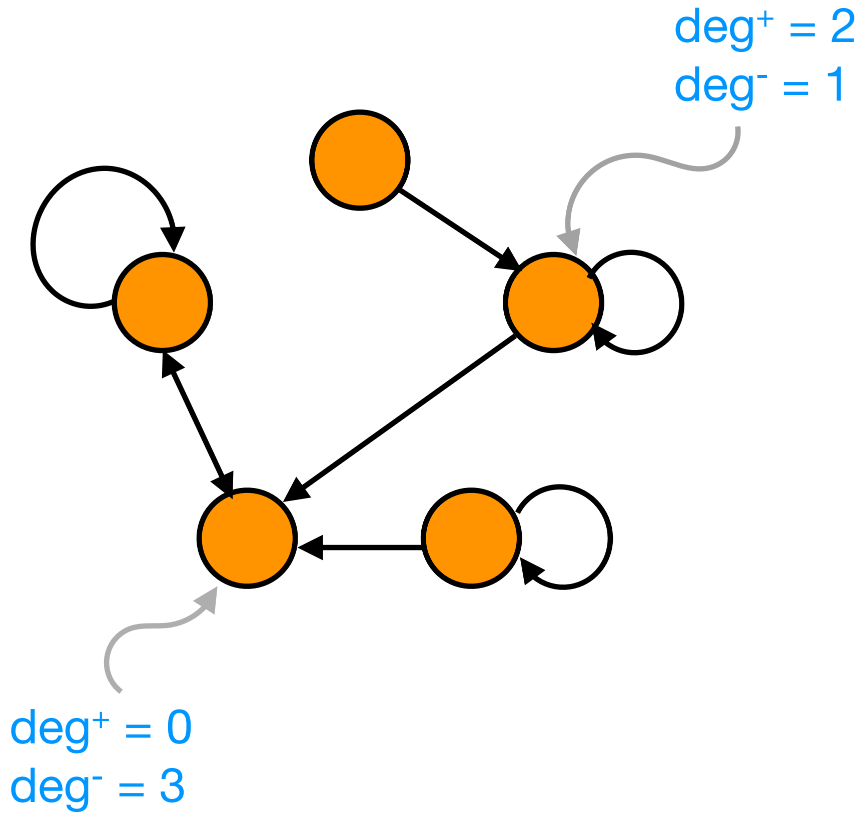
Relationship:
Directed edge between website A and website B if there is a link from website A to website B



Degrees (again)

pp. 629

- The *in-degree* $\deg^-(v)$ of a node v is the number of edges ending in the node; the *out-degree* $\deg^+(v)$ is the number of edges starting at the node.
- Formally:
  - $\deg^+(u) = |\{(u,v) \in E\}|$
  - $\deg^-(u) = |\{(v,u) \in E\}|$



Simple Degree Formulas

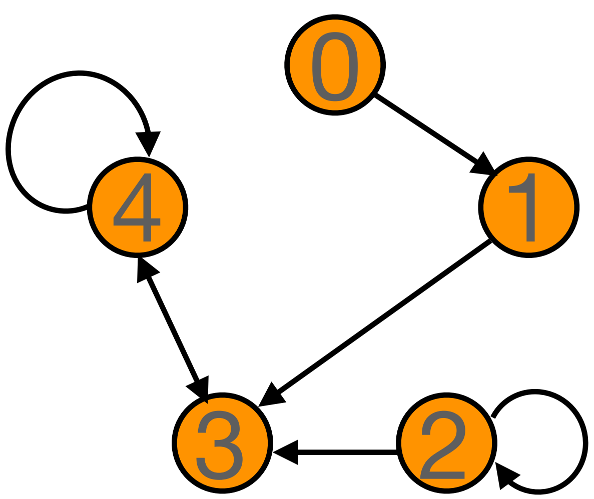
pp. 629-630

- $G = (V,E)$ directed graph. Then
  - Sum of in-degrees = sum of out-degrees = $|E|$.
    - Counting in-degrees counts every edge exactly once (every edge has exactly one destination).
    - Same for out-degrees.

Adjacency Matrix

pp. 644-646

- G = (V,E) directed graph, $V=\{v_1,\ldots,v_n\}$. An *adjacency matrix* for G is an n x n-*matrix* A=(a_{ij}) such that
  - $a_{ij} = 1$ if $(v_i,v_j) \in E$, and $a_{ij} = 0$ otherwise.
- Note that the adjacency matrix depends on the ordering of the elements of V (hence is not unique).



	0	1	2	3	4
0	0	1	0	0	0
1	0	0	0	1	0
2	0	0	1	1	0
3	0	0	0	0	1
4	0	0	0	1	1

Sum of entries in row i is the out-degree of node v_i

Matrix is not symmetric in general

Sum of entries in column i is the in-degree of node v_i