Numerical experiments to design codes on Grassmannians

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When? Algo Day – September 30th, 2009

What is a Grassmanian ?

Definition

Fix integers $d \leqslant n$.
The set of all d-dimensional vector subspaces of $\mathbb{R}^n$ is called the Grassmannian and denoted $\mathscr{G}_{n,d}$.

Properties

Any element can be represented by a $n \times d$ matrix X, unique up to a left multiplication by $\mathbf{GL}_d(\mathbb{R})$.

- It is a *variety* ($\rightsquigarrow$ we can do geometry).
- It is an *homogeneous space* ($\rightsquigarrow$ under the action of $\mathbf{GL}_n(\mathbb{R})$, it looks everywhere the same).

Where coding theory comes in

The setting

Multiple antennas.

Model :

$$Y = HX + W \qquad \text{with } X = \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix}$$

Non coherent case : H is not known . Up to noise, the receiver can only “see” the space spanned by u_1, $u_2 \ldots u_d$.

A mathematical problem

Definition For $x, x' \in \mathscr{G}_{n,d}$, define the product distance by

$$d_\Pi(x, x') = \det(\mathrm{I}_n - p_x \circ p'_x)$$

with $p_x = x(x^\dagger x)^{-1} x^\dagger$ the orthogonal projector on x.

Problem Given n, d and N, find $C \subseteq \mathscr{G}_{n,d}$ such that

- $|C| = N$
- $\min_{x,x' \in C} d_\Pi(x, x')$ be as big as possible.

A first attempt

Goal

Find numerically good configuration, try to identify them, detect families.

cf. Work by Conway, Hardin & Sloane, *Packing Lines, Planes, etc., Packings in Grassmannian Spaces*, Experimental Mathematics, Vol. 5 (1996), 139-159, for other distances on $\mathscr{G}_{n,d}$

We focus on $n = 4$, $d = 2$.

Method

- Pick randomly N points C of $\mathscr{G}_{n,d}$.
- Imagine that a "force" separate them if the distance is to small by trying to minimise a potential

$$\mathcal{P}(C) = \sum_{x,x' \in C} \frac{1}{(d_\Pi(x,x') - \alpha)^k}$$

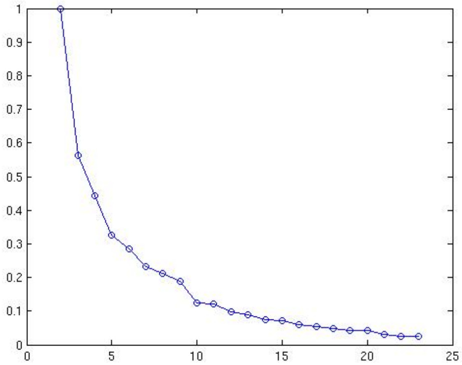
(for some parameter α, k to adjust).

- Try to improve the minimum distance by small moves.

The Hooke - Jeeves algorithm

- Start with a configuration C
- [Exploration move] Sample the potential function $\mathcal{P}$ successively on each direction (in the tangent space) by a step Δ to find a better configuration.
- [Pattern move] If the exploratory move gives a new configuration, use this direction to find a new configuration and repeat.
- Else, reduce the step Δ.
- Stop when Δ is small.

Numerical result for d_Π



How to visualise the configuration (I)

- Compute the principal angles,
  i.e. find an orthonormal basis (u_1, u_2) of $x \in C$ and an orthonormal basis (u'_1, u'_2) such that (u_1, u_2) has maximal angle in $[0, \frac{\pi}{2}]$.
  By the way, $d_\sqcap(x, x') = \sin(u_1, u'_1)^2 \sin(u_2, u'_2)^2$
  Are they nice values ? e.g. 90°, 45°, 30° ...

How to visualise the configuration (II)

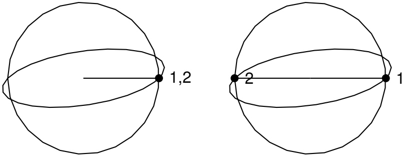
- Use binocular code,
  Represent a plane $x = \langle u_1, u_2 \rangle$ of $\mathscr{G}_{4,2}$ by the rotation $\alpha = p_x - p_{x^\perp}$ in quaternions.

Property There exists ℓ, r purely imaginary quaternions such that $\alpha = x \mapsto \bar{\ell} x r$. The pair $\pm(\ell, r)$ is unique up to sign and equal to $\pm(u_1\overline{u_2} - u_2\overline{u_1}, \overline{u_2}u_1 - \overline{u_1}u_2)$.
Thus we can visualise any code C as a subset of $S^2 \times S^2$.
Given (ℓ, r) and (ℓ', r) in $S^2 \times S^2$, the principal angles more or less are given by

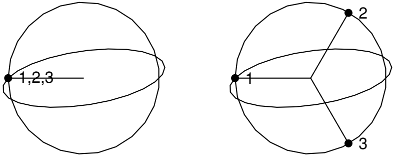
$$\frac{1}{2}\left(\widehat{(\ell, \ell')} \pm \widehat{(r, r')}\right).$$

Results

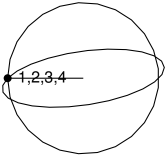
- $N = 2 : C = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$
  Principal angles $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$
  $d_{\text{min}} = 1$

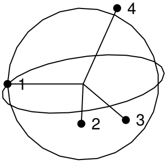


- $N = 3 :$
  Principal angles : $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$
  $d_{\text{min}} = \sin^4 \frac{\pi}{3} = \frac{9}{16}.$



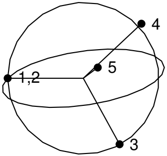
- $N = 4$
  Principal angles : (θ, θ) with $\theta = \arcsin \frac{\sqrt{6}}{3}$.

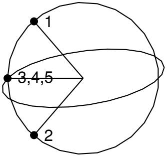




$d_{\min} = \frac{4}{9}$

- $N = 5$
  Principal angles : $(49.1^\circ, 49.1^\circ)$ or $(84.2^\circ, 35.1^\circ)$
  $d_{\min} = \frac{241}{144} - \frac{7\sqrt{3}}{9} \simeq 0.3265$





- $N = 6$ Principal angles $(47^\circ, 47^\circ)$ or $(90^\circ, 32^\circ)$
  Minimal distance : $\frac{17-4\sqrt{13}}{9} \simeq .2864$