Suppose the number of checks per supersymbol is 2.

Ripple Probability

Suppose we are in state (c, r, u) and we pick a random supersymbol z for decoding. What is the probability that a ripple element becomes useless after the decoding step? It is given by

$$p_R(u) = \Pr[E_{0,u-1} | E_{R,u}] = \frac{\Pr[E_{0,u-1} \wedge E_{R,u}]}{\Pr[E_{R,u}]}$$

where $E_{0,u-1}$ is the event that a random supersymbol is of degree 0 after the decoding step and $E_{R,u}$ is the probability that a random supersymbol is in the ripple before decoding.

We could apply the previously obtained formulas:

$$\Pr[E_{0,u-1} \wedge E_{R,u}] = \sum_{i,j,=1}^{r} \sum_{d,d'} \Omega_d \Omega_{d'} \binom{d}{i} \binom{d'}{j} \frac{\begin{bmatrix} u \\ j \end{bmatrix} \begin{bmatrix} k-u \\ d'-j \end{bmatrix}}{\begin{bmatrix} k \\ d' \end{bmatrix}} \frac{\begin{bmatrix} j \\ i \end{bmatrix} \begin{bmatrix} k-u \\ d-i \end{bmatrix}}{\begin{bmatrix} k \\ d \end{bmatrix}},$$

$$\Pr[E_{R,u}] = \sum_{i=1}^{r} \frac{\sum_d \Omega_d \binom{d}{i} \begin{bmatrix} u \\ i \end{bmatrix} \begin{bmatrix} k-u \\ d-i \end{bmatrix}}{\begin{bmatrix} k \\ d \end{bmatrix}}.$$

BUT note the following: picking a random ripple element z' of reduced degree j for decoding is EQUIVALENT to picking j input symbols uniformly at random and recovering them in one shot. This is because the encoding was uniformly random and the choice of symbols to recover up till this point in the decoding was also uniformly random.

Let z be a uniformly chosen output supersymbol of initial degree d. The probability that it is in the ripple when u symbols are undecoded is

$$\begin{aligned}\Pr[E_{R,u}] &= \Pr[z \in R \text{ at } u] \\ &= \Pr[\deg(z) = 1 \text{ at } u] + \Pr[\deg(z) = 2 \text{ at } u] \\ &= \frac{d\begin{bmatrix} k-u \\ d-1 \end{bmatrix} + \binom{d}{2}\begin{bmatrix} u \\ 2 \end{bmatrix}\begin{bmatrix} k-u \\ d-2 \end{bmatrix}}{\begin{bmatrix} k \\ d \end{bmatrix}}.\end{aligned} \tag{1}$$

Now to the calculation of $\Pr[E_{0,u-1} \wedge E_{R,u}]$. For this we need to condition on the number of input symbols recovered at this decoding step. Let $E_{1,u}$ (resp., $E_{2,u}$) be the event that a randomly picked output supersymbol has reduced degree 1 (resp., 2) when u symbols are undecoded. We know that the fraction of ripple elements that have reduced degree 1 is given by

$$\begin{aligned}p_1 &:= \Pr[E_{1,u} | E_{R,u}] \\ &= \frac{\Pr[E_{1,u} \wedge E_{R,u}]}{\Pr[E_{R,u}]} \\ &= \frac{\Pr[E_{1,u}]}{\Pr[E_{R,u}]}\end{aligned} \tag{2}$$

where $\Pr[E_{R,u}]$ is given by equation (1) and

$$\Pr[E_{1,u}] = \sum_d \Omega_d \frac{du\begin{bmatrix} k-u \\ d-1 \end{bmatrix}}{\begin{bmatrix} k \\ d \end{bmatrix}}. \tag{3}$$

Similarly, the fraction of ripple elements that have reduced degree 2 is given by

$$p_2 := \Pr[E_{2,u} | E_{R,u}] = \frac{\Pr[E_{2,u}]}{\Pr[E_{R,u}]}, \tag{4}$$

where

$$\Pr[E_{2,u}] = \sum_d \Omega_d \frac{\binom{d}{2} u(u-1)\begin{bmatrix} k-u \\ d-2 \end{bmatrix}}{\begin{bmatrix} k \\ d \end{bmatrix}}. \tag{5}$$

Now

$$\Pr[E_{0,u-1} \wedge E_{R,u}] = \sum_{\ell=1,2} \Pr[z \in R \text{ and } z \text{ leaves } R | \ell \text{ symbols recovered}] \cdot \Pr[\ell \text{ symbols recovered}]$$

$$\begin{aligned} &= p_1 \Pr[\deg z = 1 \text{ and } z \text{ leaves } R | 1 \text{ symbol recovered}] \\ &\quad + p_2 \big(\Pr[\deg z = 1 \text{ and } z \text{ leaves } R | 2 \text{ symbols recovered}] \\ &\qquad + \Pr[\deg z = 2 \text{ and } z \text{ leaves } R | 2 \text{ symbols recovered}] \big), \end{aligned}$$

where z is an output symbol chosen uniformly at random. Let

$$\begin{aligned} a &:= \Pr[\deg z = 1 \text{ and } z \text{ leaves } R | 1 \text{ symbol recovered}] \\ b &:= \Pr[\deg z = 1 \text{ and } z \text{ leaves } R | 2 \text{ symbols recovered}] \\ c &:= \Pr[\deg z = 2 \text{ and } z \text{ leaves } R | 2 \text{ symbols recovered}]. \end{aligned}$$

Then

$$\begin{aligned} a &= \Pr[\deg z = 1 \text{ and } z \text{ leaves } R | 1 \text{ symbol recovered}] \\ &= \Pr[\deg z = 1 | 1 \text{ symbol recovered}] \cdot \Pr[z \text{ leaves } R | \deg z = 1 \text{ and } 1 \text{ symbol recovered}] \\ &= \Pr[\deg z = 1] \cdot \Pr[z \text{ leaves } R | \deg z = 1 \text{ and } 1 \text{ symbol recovered}] \\ &= \Pr[E_{1,u}] \cdot \frac{1}{u}, \end{aligned}$$

where $\Pr[E_{1,u}]$ is given by equation (3). Similarly,

$$\begin{aligned} b &= \Pr[\deg z = 1 \text{ and } z \text{ leaves } R | 2 \text{ symbols recovered}] \\ &= \Pr[E_{1,u}] \cdot \Pr[z \text{ leaves } R | \deg z = 1 \text{ and } 2 \text{ symbols recovered}] \\ &= \Pr[E_{1,u}] \cdot \frac{2}{u}, \end{aligned}$$

and

$$\begin{aligned} c &= \Pr[\deg z = 2 \text{ and } z \text{ leaves } R | 2 \text{ symbols recovered}] \\ &= \Pr[E_{2,u}] \cdot \Pr[z \text{ leaves } R | \deg z = 2 \text{ and } 2 \text{ symbols recovered}] \\ &= \Pr[E_{2,u}] \cdot \frac{1}{\binom{u}{2}} \\ &= \Pr[E_{2,u}] \cdot \frac{2}{u(u-1)}, \end{aligned}$$

where $\Pr[E_{2,u}]$ is given by equation (5).

Finally, we get that

$$\begin{aligned}\Pr[E_{0,u-1} \wedge E_{R,u}] &= p_1 \Pr[E_{1,u}]\frac{1}{u} + p_2 \left(\Pr[E_{1,u}] \cdot \frac{2}{u} + \Pr[E_{2,u}] \cdot \frac{2}{u(u-1)} \right) \\ &= \frac{\Pr[E_{1,u}]^2}{\Pr[E_{R,u}]}\frac{1}{u} + \frac{\Pr[E_{1,u}]\Pr[E_{2,u}]}{\Pr[E_{R,u}]}\frac{2}{u} + \frac{\Pr[E_{2,u}]^2}{\Pr[E_{R,u}]}\frac{2}{u(u-1)}. \end{aligned} \tag{6}$$

Putting this together with the expression for $\Pr[E_{R,u}]$ given by equation (1), we get an expression for the probability that a random ripple element leaves the ripple at this decoding step:

$$\begin{aligned} p_u(R) &= \frac{\Pr[E_{0,u-1} \wedge E_{R,u}]}{\Pr[E_{R,u}]} \\ &= \frac{1}{\Pr[E_{R,u}]^2} \left(\Pr[E_{1,u}]^2 \frac{1}{u} + \Pr[E_{1,u}]\Pr[E_{2,u}]\frac{2}{u} + \Pr[E_{2,u}]^2 \frac{2}{u(u-1)} \right). \end{aligned} \tag{7}$$

Cloud Probability

We now want to calculate the probability $p_u(C)$ that a random cloud element joins the ripple at this decoding step. This is $\Pr[E_{R,u-1}|E_{C,u}]$, where $E_{R,u-1}$ is the event that a random output supersymbol is in the ripple when $u-1$ symbols are undecoded, and $E_{C,u}$ denotes the event that a random output supersymbol is in the cloud when u symbols are undecoded. We have

$$\begin{aligned} p_u(C) &= \Pr[E_{R,u-1}|E_{C,u}] \\ &= \frac{\Pr[E_{R,u-1} \wedge E_{C,u}]}{\Pr[E_{C,u}]}. \end{aligned}$$

Here $\Pr[E_{C,u}]$ is given by

$$\Pr[E_{C,u}] = 1 - \Pr[E_{0,u}] - \Pr[E_{R,u}], \tag{8}$$

where

$$\Pr[E_{0,u}] = \sum_d \Omega_d \frac{\begin{bmatrix} k-u \\ d \end{bmatrix}}{\begin{bmatrix} k \\ d \end{bmatrix}}$$

and $\Pr[E_{R,u}]$ is given by equation (1).

As for $\Pr[E_{R,u-1} \wedge E_{C,u}]$, we again condition on the number of input symbols recovered at this step to get

$$\begin{aligned} \Pr[E_{R,u-1} \wedge E_{C,u}] &= \sum_{\ell=1,2} \Pr[E_{R,u-1} \wedge E_{C,u} | \ell \text{ symbols recovered}] \cdot \Pr[\ell \text{ symbols recovered}] \\ &= p_1 \Pr[E_{R,u-1} \wedge E_{C,u} | 1 \text{ symbol recovered}] \\ &\quad + p_2 \Pr[E_{R,u-1} \wedge E_{C,u} | 2 \text{ symbols recovered}], \end{aligned}$$

where p_1 and p_2 are given by equations (2) and (4), respectively.

Now $\Pr[E_{R,u-1} \wedge E_{C,u} | 1 \text{ symbol recovered}]$ is equal to

$$\Pr[\deg z = 3 \text{ and then } 2 | 1 \text{ symbol recovered}] = \sum_d \Omega_d \frac{d\binom{d-1}{2} \begin{bmatrix} u-1 \\ 2 \end{bmatrix} \begin{bmatrix} k-u \\ d-3 \end{bmatrix}}{\begin{bmatrix} k \\ d \end{bmatrix}}.$$

As for $\Pr[E_{R,u-1} \wedge E_{C,u} | 2 \text{ symbols recovered}]$, it is given by

$$\begin{aligned} &\Pr[\deg z = 3 \text{ and then } 1 | 2 \text{ symbols recovered}] + \Pr[\deg z = 4 \text{ and then } 2 | 2 \text{ symbols recovered}] \\ &= \sum_d \Omega_d \frac{\binom{d}{2} 2 \binom{d-2}{1} (u-1) \begin{bmatrix} k-u \\ d-3 \end{bmatrix}}{\begin{bmatrix} k \\ d \end{bmatrix}} + \sum_d \Omega_d \frac{\binom{d}{2} 2 \binom{d-2}{2} \begin{bmatrix} u-1 \\ 2 \end{bmatrix} \begin{bmatrix} k-u \\ d-4 \end{bmatrix}}{\begin{bmatrix} k \\ d \end{bmatrix}}. \end{aligned}$$