Neural Association and Coding Theory

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A joint project with:

Raj K. Kumar

EPFL

algolma

In this talk...

- Motivation
- Introduction to Neural Networks
- Current problem formulation
- Proposed solution(s)
- A glimpse to future works
- Questions and discussion

Motivation

- Coding theory: finding methods to overcome the issue of noise in communications.

- Project goals:
  - Employ mathematical background of coding theory to analyze neural networks.
  - Inspire from neural operations to design codes.
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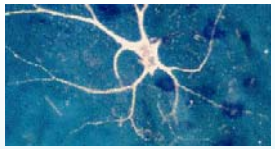
A BRIEF INTRODUCTION TO NEURAL NETWORKS

First Some Terminologies



- A Neuron is the most basic entity in the nervous system that processes information.
- Neurons also transmit information via electrical pulses, called spikes.
- The information about the outside world is encoded in the spike train:
  - usually it is encoded in the number of spikes per second (rate code),
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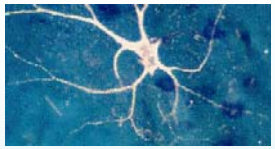

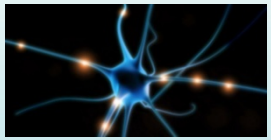

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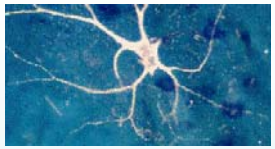

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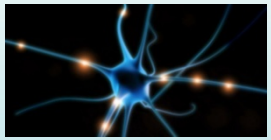

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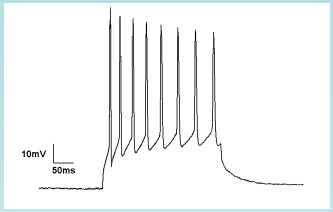

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Fired = + 1 Silent = -1

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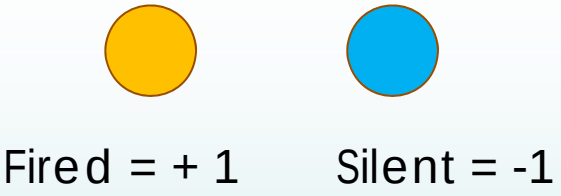
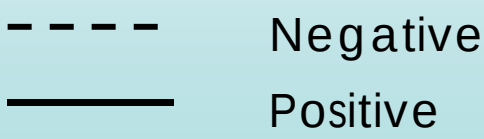
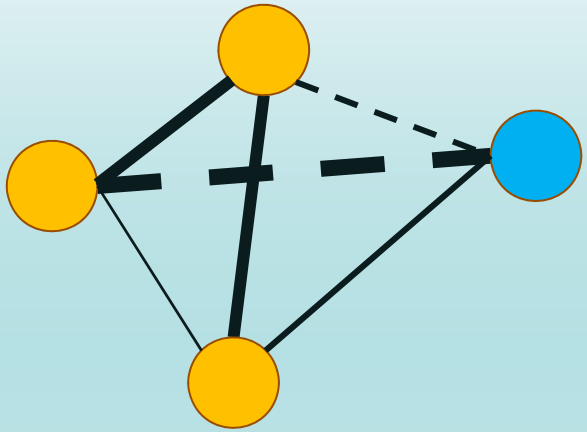


- The neural network:





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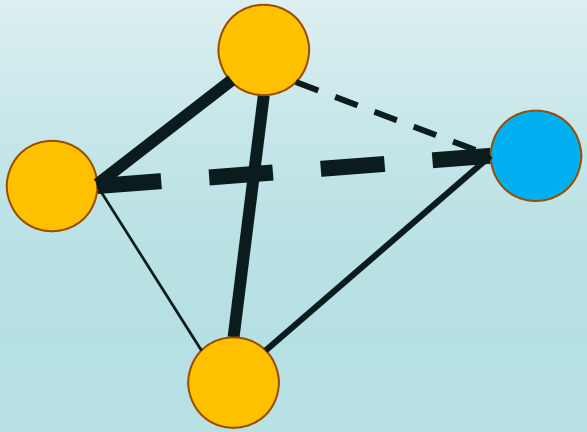
Positive



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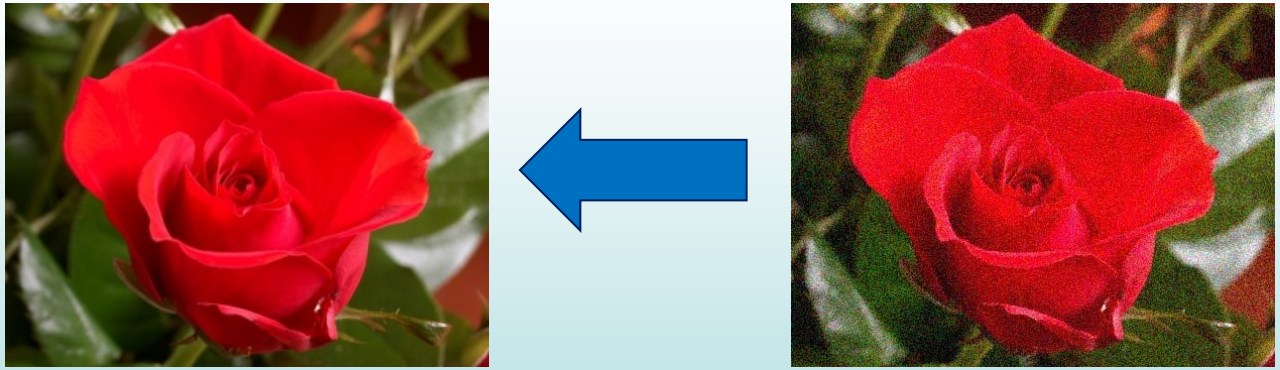

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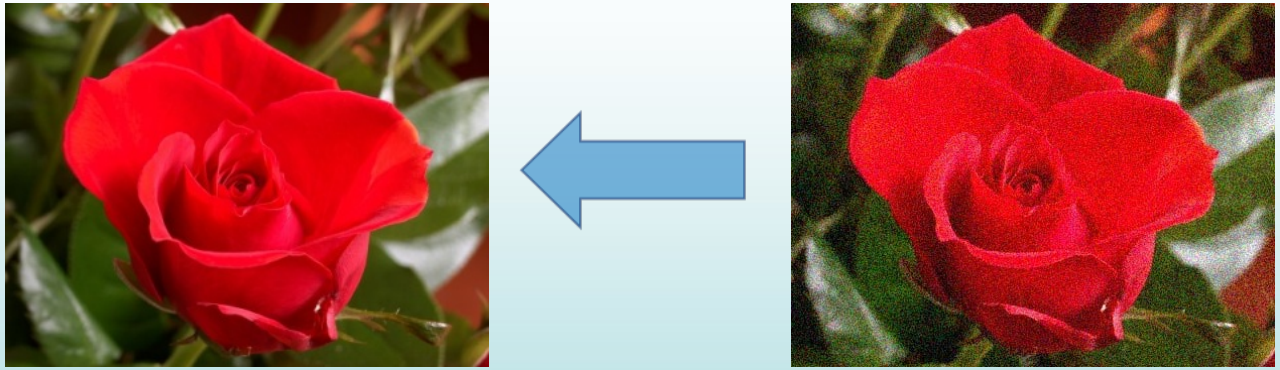

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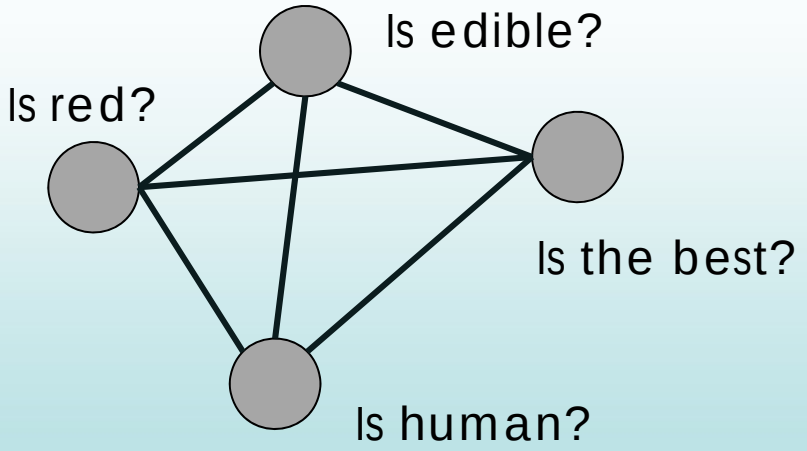

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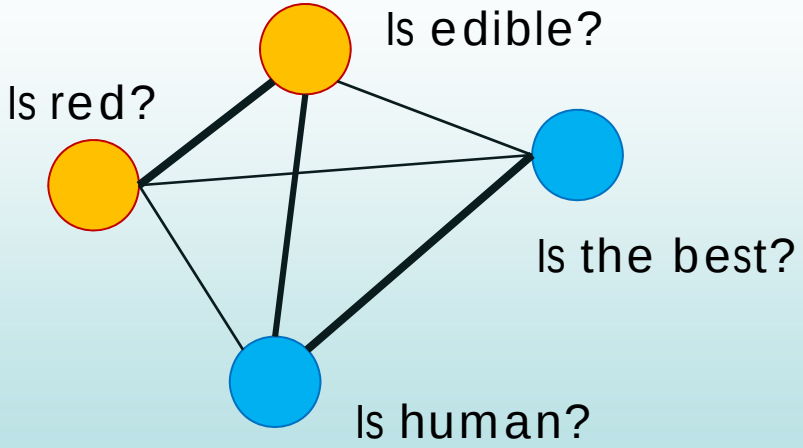

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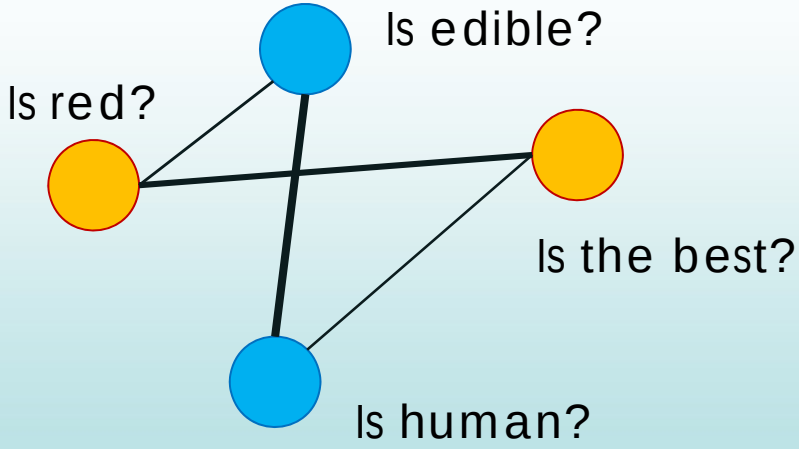

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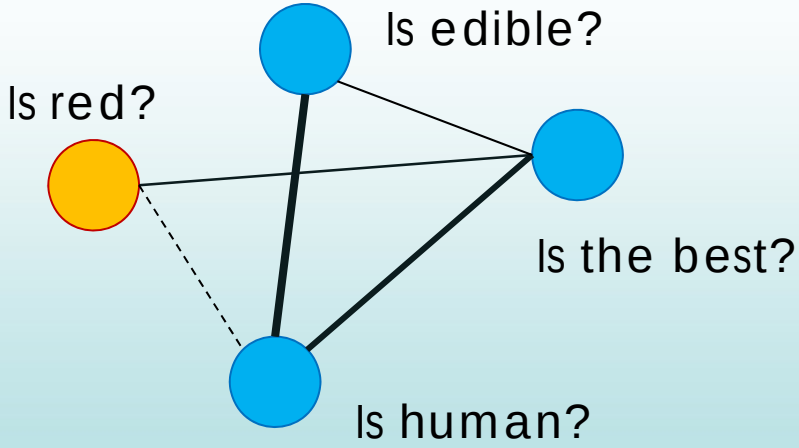

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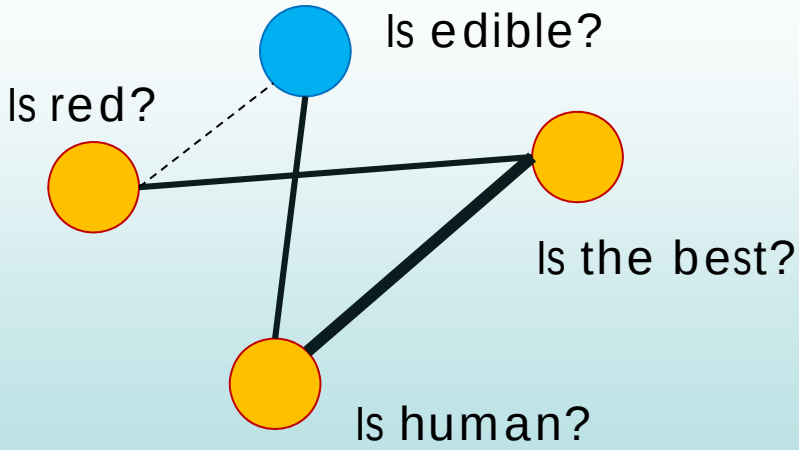

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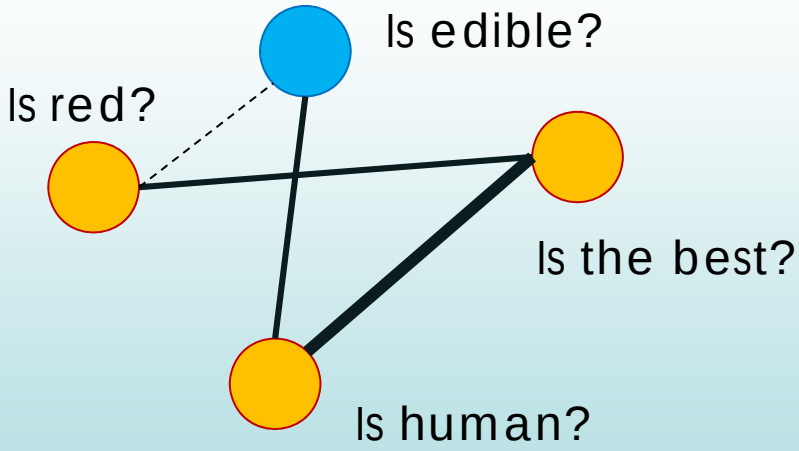

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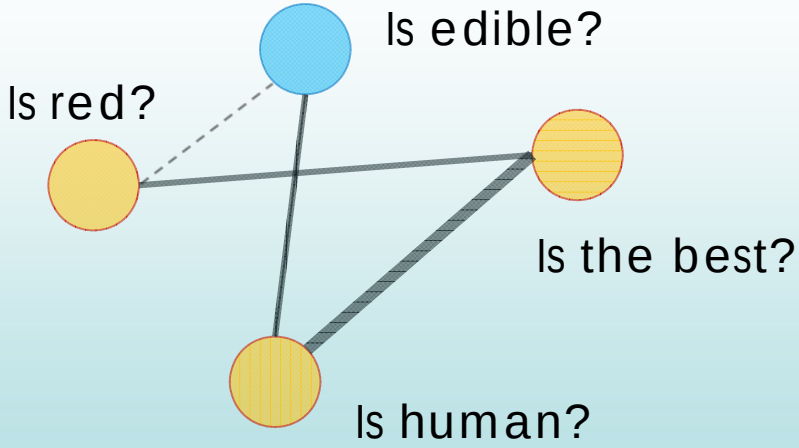

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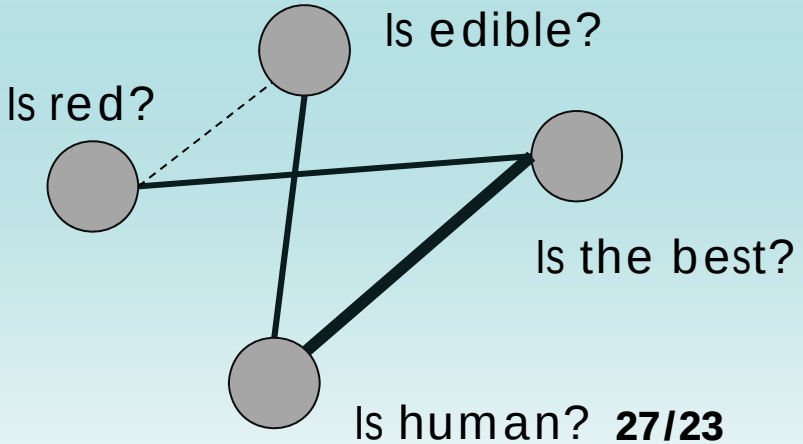

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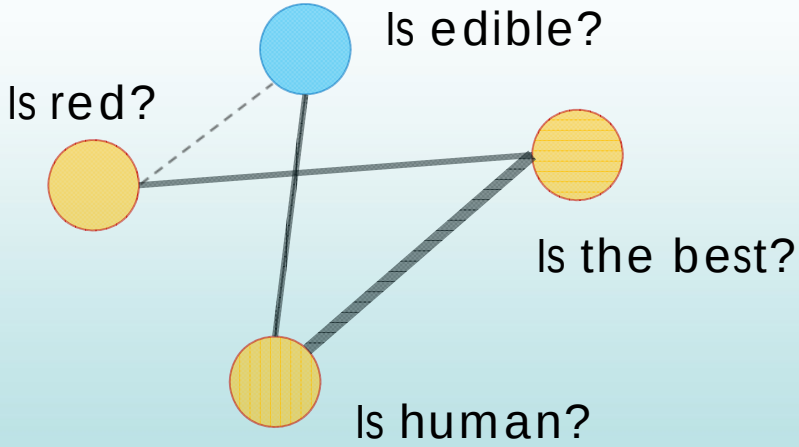

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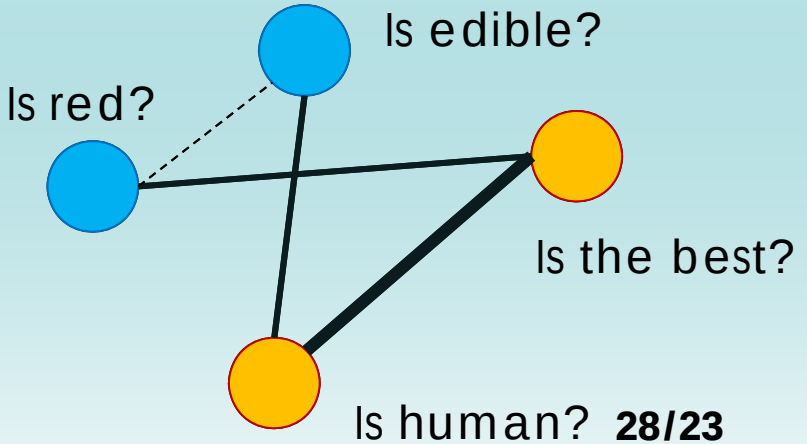

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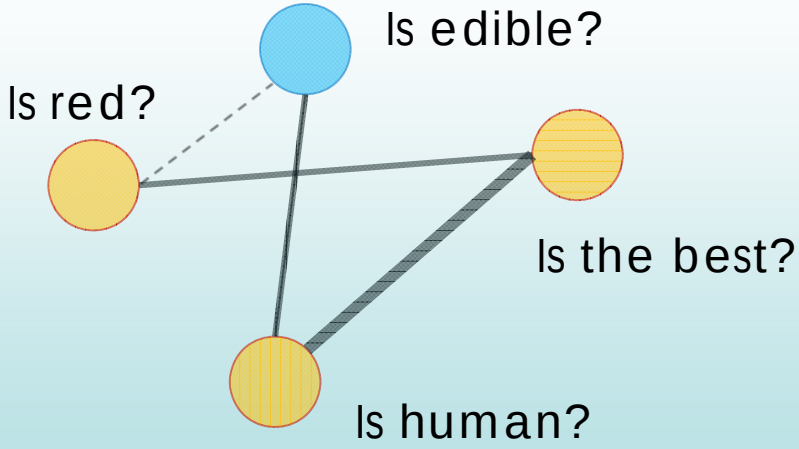

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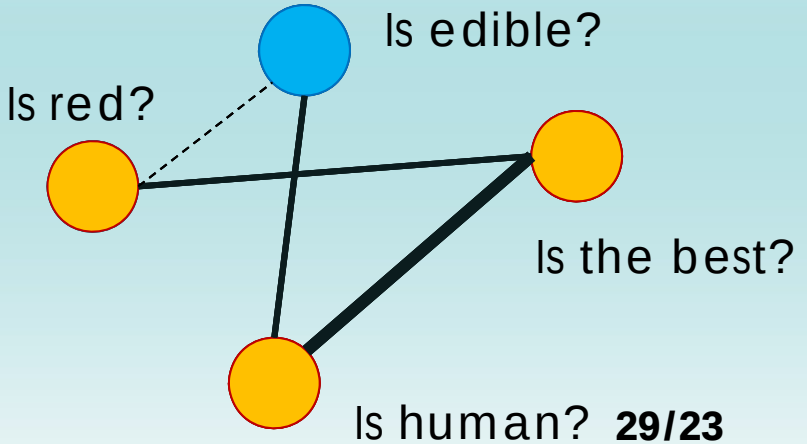

PROBLEM FORMULATION

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  - Stability
  - Noise tolerance

Hopfield Associative Memory

- Hopfield weighting rule:

$$w_{ij} = \frac{1}{n} \sum_{\mu=1}^{M} x_i^\mu x_j^\mu$$

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Vanishing *bit* error probability:	Vanishing *pattern* error probability:
$M_{max} = 0.138\ n$	$M_{max} = O(n/log(n))$

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- Random sequences → bad distance properties
- We suggest considering non-random patterns.
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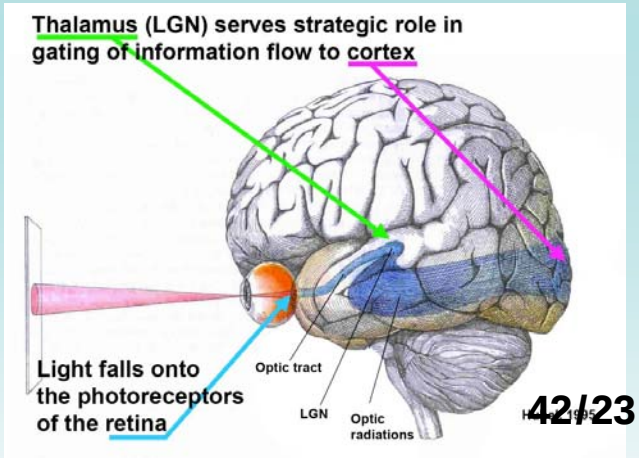
- Is it biologically meaningful?
- Could be: visual pathway is a good example.



Memorizing Patterns with Low Correlation

- First candidate: codewords of a linear code.
  - Hopfield network can not learn XOR!
- Second attempt: **Gold sequences**.
  - Gold family is a set of binary vectors with optimal correlation properties.
- Why low correlation?

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$$= x_i^p + \frac{1}{n} \sum_{\mu=1 \neq p}^{M} x_i^\mu \sum_{j=1}^{n} x_j^p x_j^\mu$$

Undesired correlation term

Memorizing Patterns with Low Correlation (contd.)

- With original Hopfield rule and Gold sequences, we will have:
  - ✓ Stability
  - ✗ But not error correction
- Solution: modify the weighting rule

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Some Results

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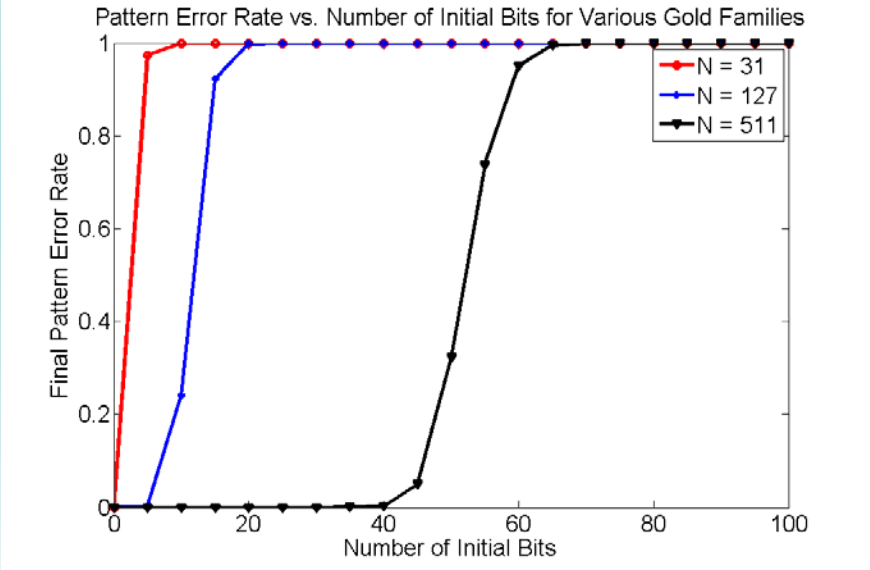

A GLIMPSE TOWARD FUTURE

Lattice Codes & Neural Associative Memory

- Two main differences between a neural graph and that of an LDPC code:
  - Neural update rule
  - GF(2) operation
- The latter can be resolved by considering codes on real fields, like lattice codes.
- The trick is to consider spike train as the neurons state.

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The Model

- Everything the same as before.
- Except the update rule: $s_i = f(\sum_{j=1}^{N} w_{ij} s_j)$
- **Constraints**:

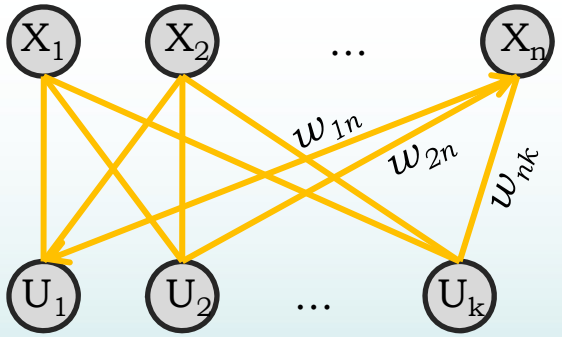
1) Neural states must be positive integers
   - Although, we might have a trick for this in our sleeves as well!
2) Reasonable weights

Some Preliminary Works

- Asymmetric

  Hetero-association:

  - Lattice code: x = u.G



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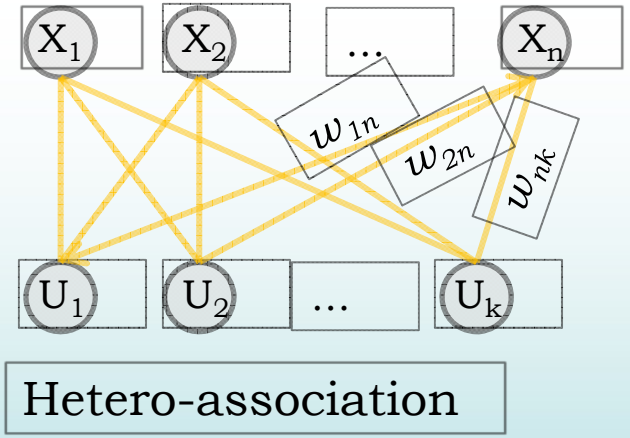

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Some Preliminary Works (contd.)

- **Recall phase**:

  Retrieve ***u*** from a noisy version of ***x***

- If we pick W_b such that $G.W_b = I$ then the input sum to z_i is:

$$h_i = (x + e)W_b = u + eW_b = u + e'$$

- So we need **G** such that e' is small.

Some Results

- Message symbols: vectors of length ***k*** with elements either being ***up*** and ***down***.
- **G**: a full rank k×n random matrix with large norm.
- Neural output
  - At code neuron:
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- Such a simple scheme can result in exponential capacities.

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- Multi-layered neural networks:
  - Hopfield networks can not learn XOR
  - Boltzmann machines can!

Thank You!

Any

Questions?