EXPONENTIAL PATTERN RETRIEVAL CAPACITY WITH NON-BINARY ASSOCIATIVE MEMORY

Amir Hesam Salavati

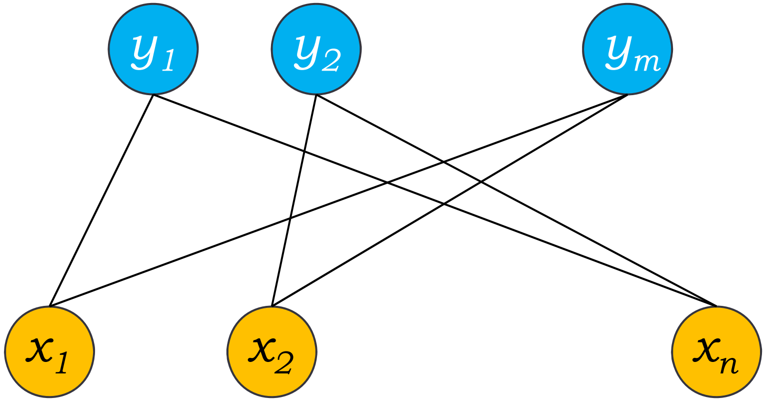
Joint work with: Raj K. Kumar Amin Shokrollahi

In This Talk...

- The problem
- Our proposed solution
  - Intuition
  - Details
  - Results
- Work in progress
- Conclusions and final remarks

The Problem in a Nutshell

- For a coding theorist:
  - Given: A *parity check graph*.
  - Required: A "*simple*" message passing decoding algorithm with restrictions on decoding nodes.



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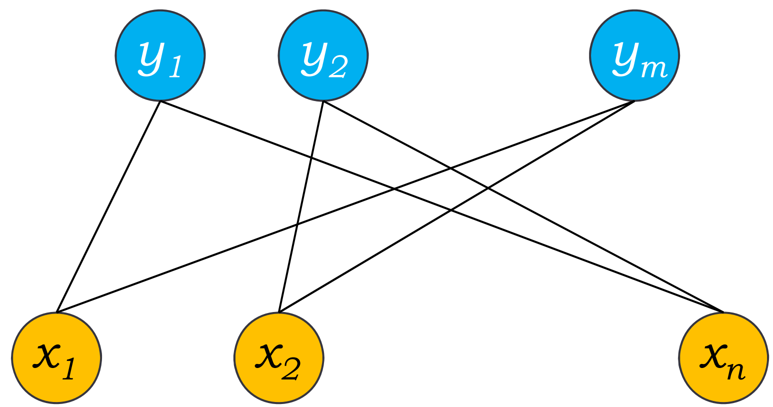

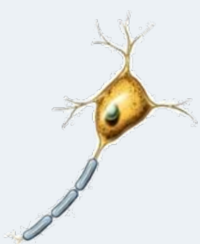

- For a neuroscientist:
  - Given: A *hetero-associative* neural network.
  - Required: A method to increase the storage capacity.

Why Is It Interesting?

- Associative memory problem: store a set of random binary patterns of length *n* reliably. Later, return the closest stored pattern in response to a noisy query.

WHY IS IT INTERESTING?

Introduction

- Hopfield, 1982
- Amit et al., 1985
- McEliece et al. 1987
- Komlos et al., 1993
- Muezzinoglu et al., 2003

- Associative memory problem: store a set of random binary patterns of length *n* reliably. Later, return the closest stored pattern in response to a noisy query.
- For the past three decades, best neural realizations yield pattern retrieval capacities linear in *n*.

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- For the past three decades, best neural realizations yield pattern retrieval capacities linear in n.
- For similar structures, we have exponential "pattern retrieval" capacities in coding theory.

Solution Idea

- The reason for the gap? Might be the pure randomness requirement.
- What if we only focus on memorizing structured patterns?
  - Better distant properties.

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Introduction

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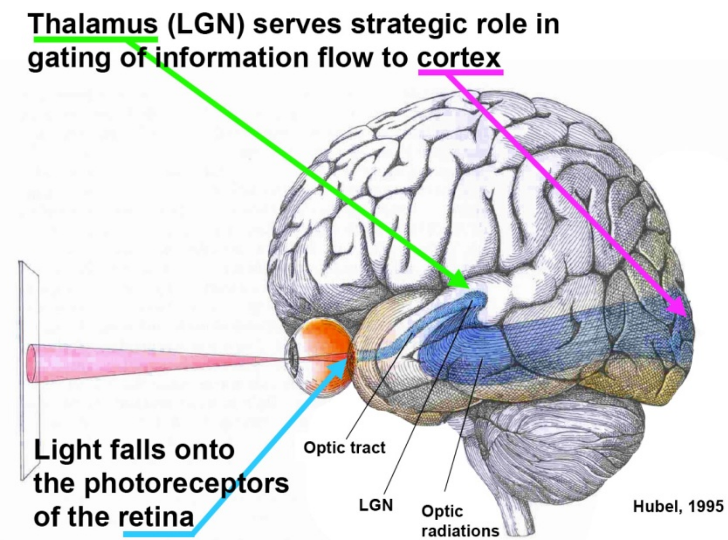
- Successful recent attempts to increasing storage capacities using *structured* patterns.
- It seems biologically relevant as well.



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Our Suggested Solution

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  - An algorithm with simple decoding nodes.
  - Message passing over non-binary codes with expander parity check graphs and majority voting broadcast nodes.

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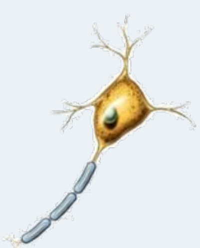
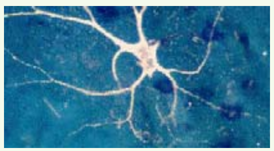


- In coding theoretical terminology:
  - An algorithm with simple decoding nodes.
  - Message passing over non-binary codes with expander parity check graphs and majority voting broadcast nodes.
- Rephrased in neuroscience parlance:
  - Only store patterns that satisfy some constraints.
  - Constraints from pre-processing stages in brain or outside world.
  - Constraints will help in dealing with noise.

Model and Method

NEURAL NETWORKS



- Neuron: a basic processor in the nervous system.

Solution

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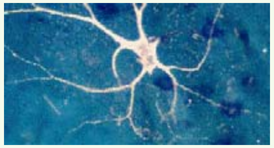

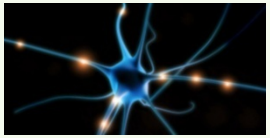

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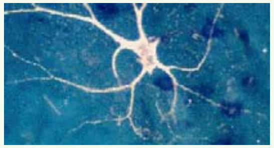
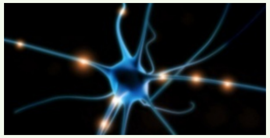




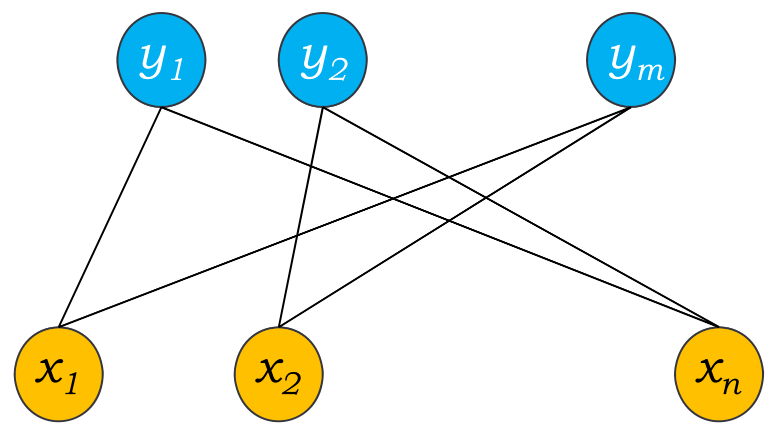
Solution

- Neuron: a basic processor in the nervous system.
- Neurons communicate via spikes.
- Neurons can:
  - Compute a linear sum (count the spikes they receive).
  - Transmit a spike train based on this sum.
  - What they transmit goes to all their neighbors (broadcast system).

MODEL

Solution

- A bipartite graph with ***n*** *pattern nodes* and ***m*** *constraint nodes*.
  - Nodes represent neurons.
  - Link weights are 0 or 1.



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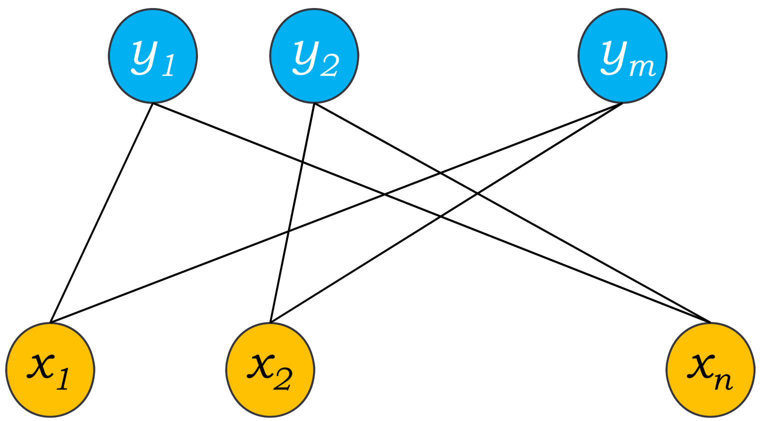

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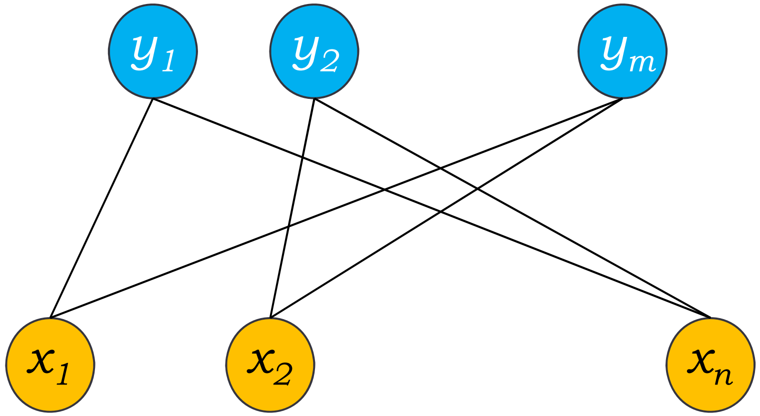

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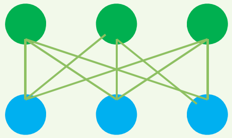

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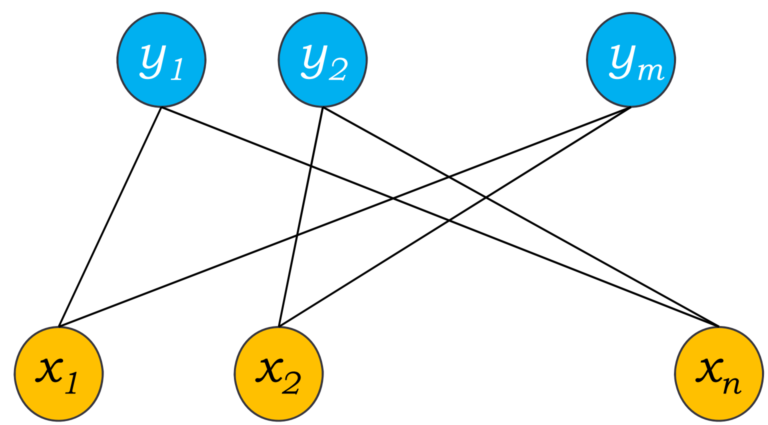

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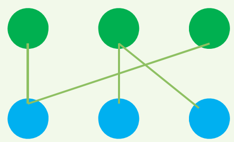

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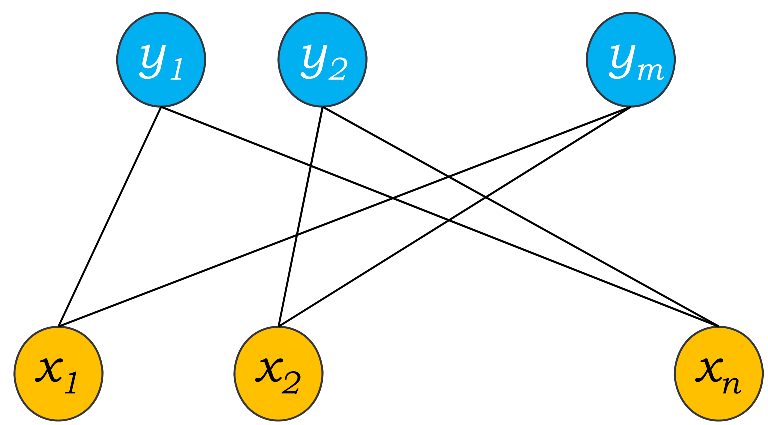

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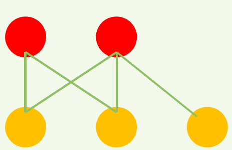

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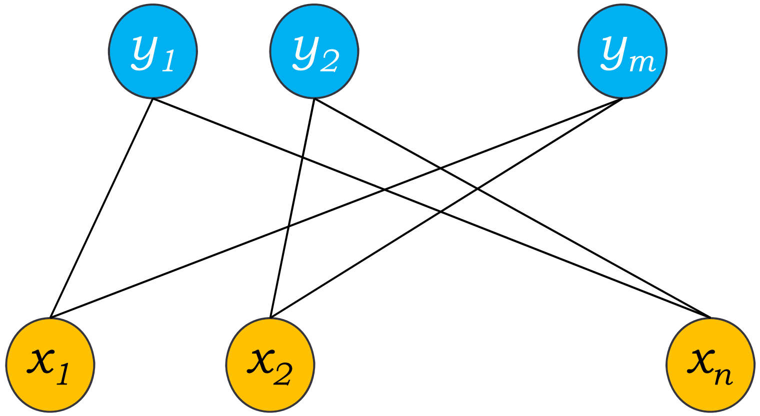

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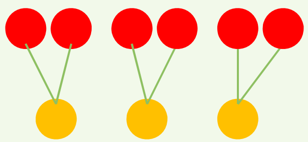

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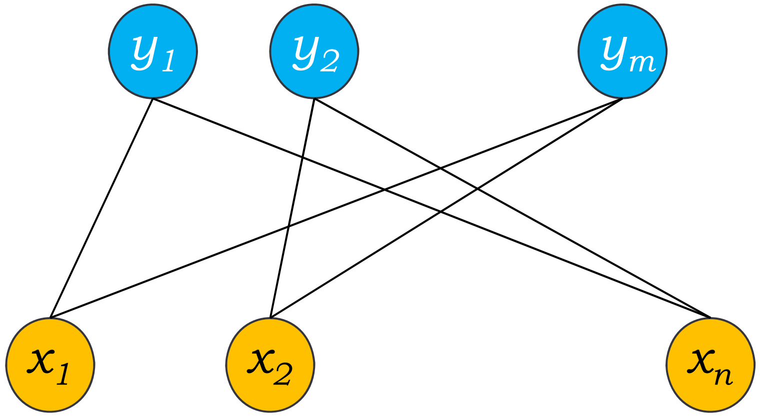
- *Sparse* and *expander*.
- Non-binary neurons
  - Output of a node is the firing rate of the neuron.



Solution

How It Works

1. Initialization
2. Iterative update
3. Convergence

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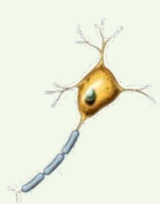

- Iterative update of nodes states according to some update rule.
- Hopefully, we can correct errors in the input.
- The structure is the same as LDPC parity check matrices except for:
  - Real field operations.
  - Simple broadcast nodes.
- The structure is similar to a *hetero-associative* memory.
- State of each node = short term firing rate of neurons.

Design Goal

- Given: a set of integer-valued vectors of length *n*.
- Required:
  - These patterns are stable states of the network.
  - Some noise be tolerated.

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  - The nodes/neurons update rule.
- For the moment, we assume H is given and only address the neural update rule.

The Algorithm: Intuition

- We are interested in patterns that satisfy certain number of constraints.
  - Different from the widely used assumption of memorizing any set of purely random patterns.

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- Given a correct pattern x^{μ}, all constraints are satisfied.
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Stability

THE ALGORITHM: INTUITION (CONTD.)

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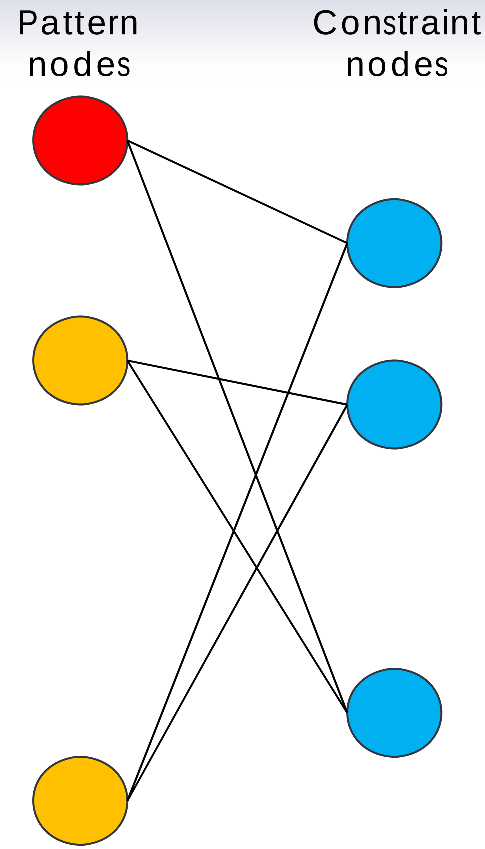
- If the network is given x^{μ} + noise:



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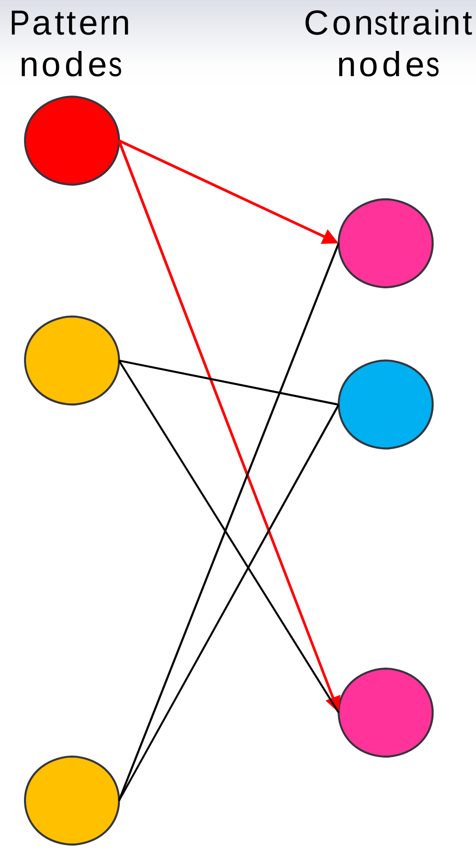

The Algorithm: Intuition (contd.)

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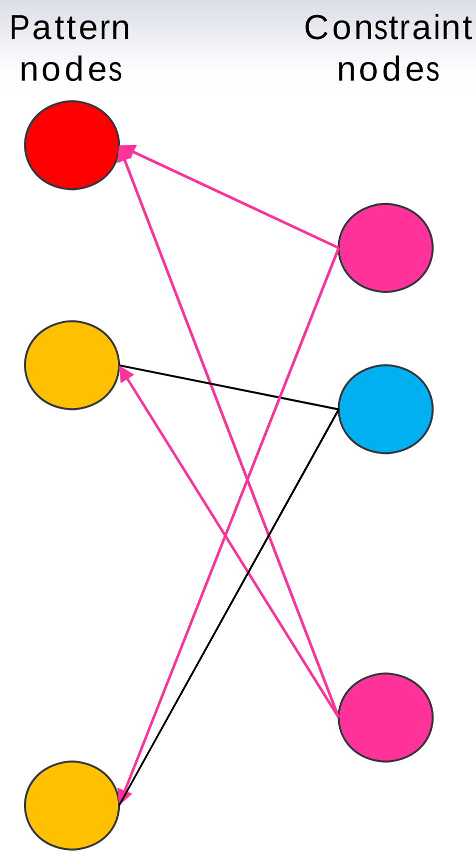

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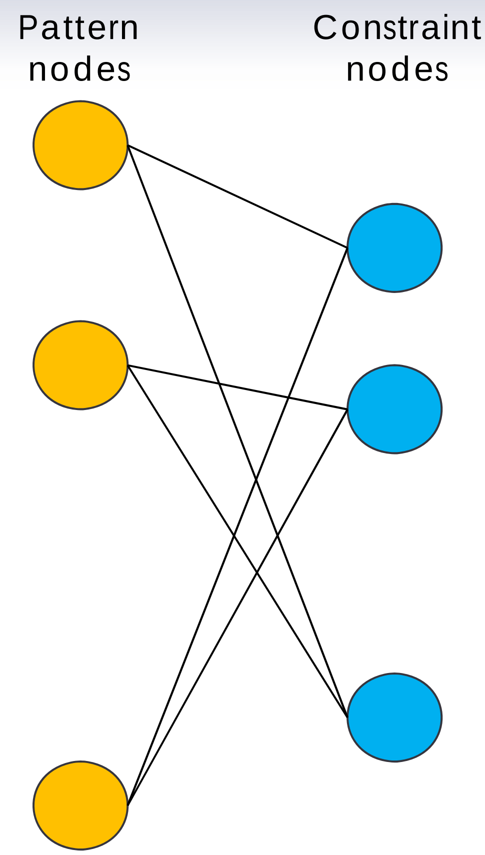

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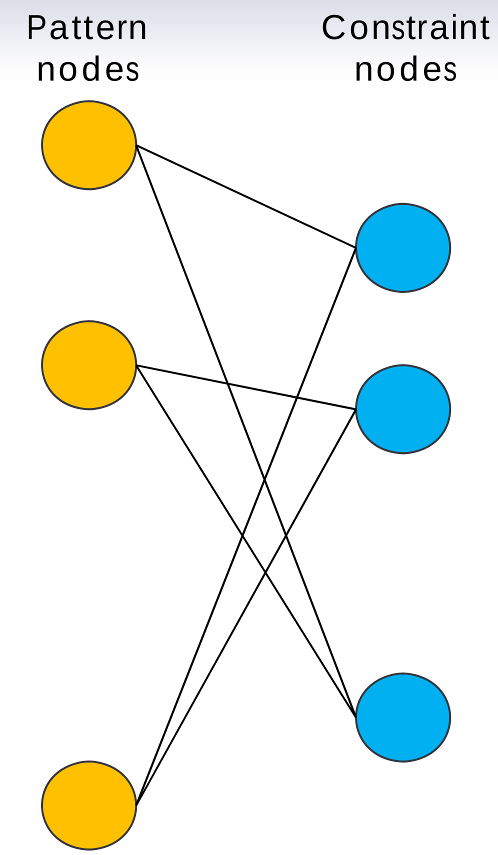

The Algorithm: Intuition (contd.)

Solution

Similar in nature to Sipser & Spielsman's expander codes.

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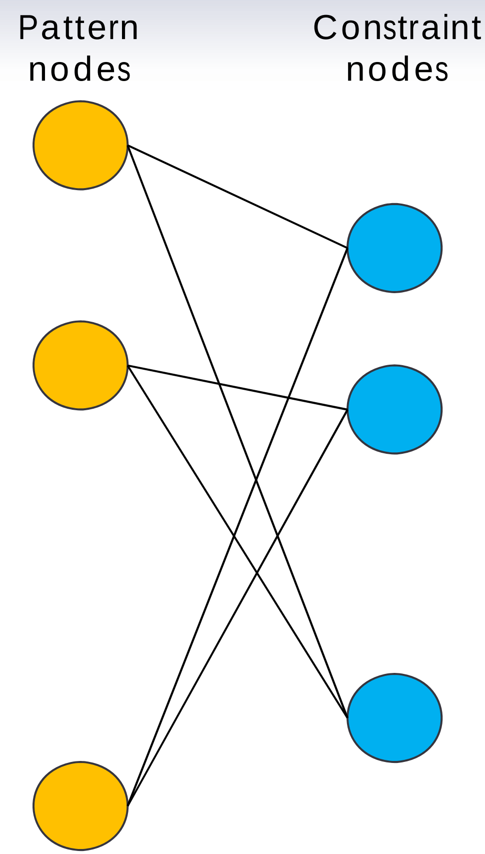


THE ALGORITHM: DETAILS

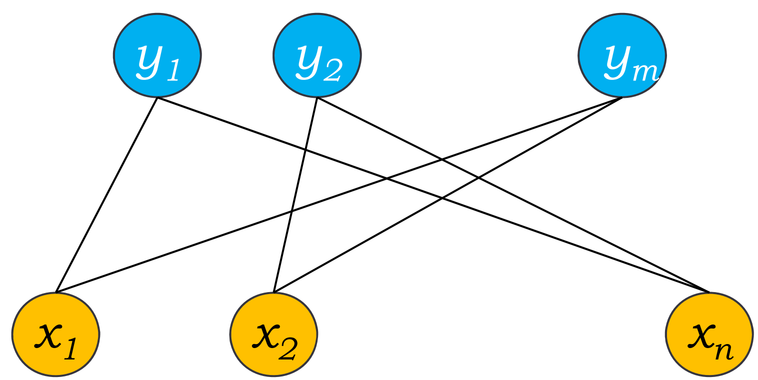
Solution

H: m×n 0/1 matrix.

x: n×1 integer vector.

b: m×1 integer vector.

- The constraint satisfaction problem: $Hx=b$



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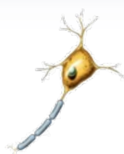
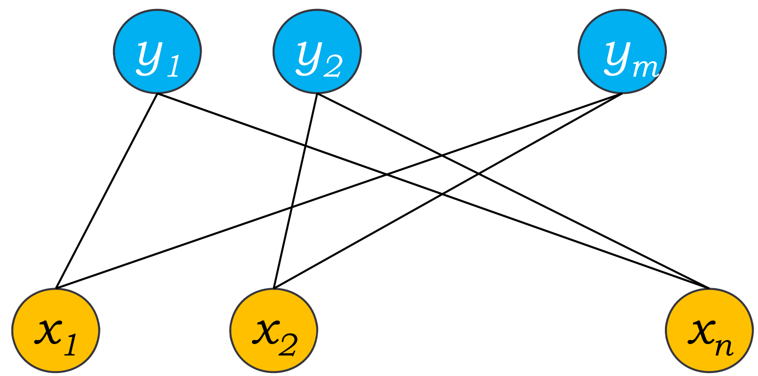
  - Firing rate of a number of pattern nodes should always add up to some value.





The Algorithm: Details

Solution

H: $m \times n$ 0/1 matrix.

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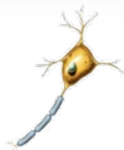
$h_i = \Sigma\, H_{ij}\, x_j$

- The constraint satisfaction problem: $Hx=b$



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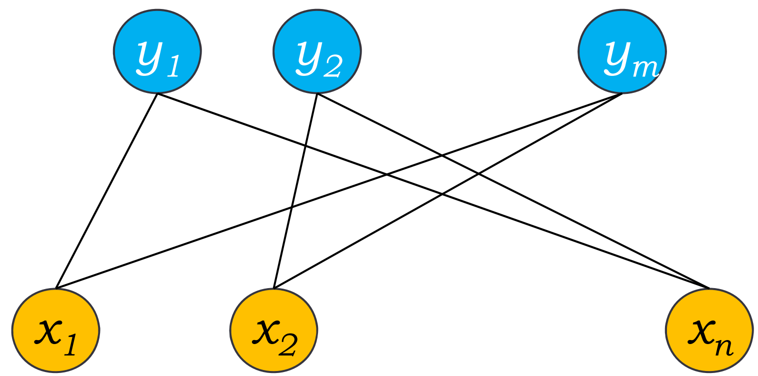

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$$y_i = \begin{cases} 1, & h_i < b_i \\ 0, & h_i = b_i \\ -1, & \text{otherwise} \end{cases}$$

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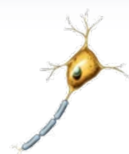

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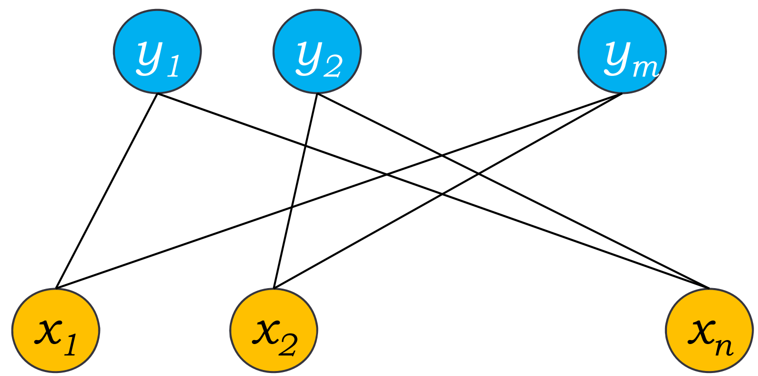

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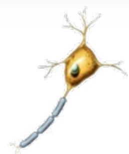

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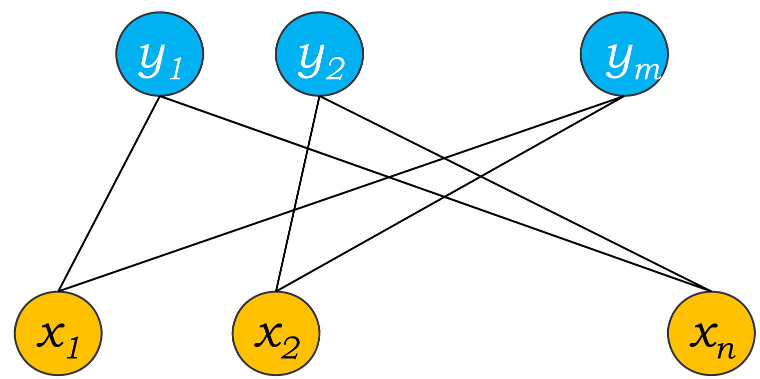

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  - Winner-take-all
  - Bit-flip

THE ALGORITHM: DETAILS (CONTD.)

Solution

The Algorithm: Details (contd.)

$\mathrm{H}x^{\mu} = b$

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Solution

- Noise free case: If x^{μ} is given, no constraint node fires.

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- If x^{μ} + noise is given:

$$g_j = \frac{\sum_{i=1}^{m} H_{ij} y_i}{d_p}$$

  - g_j is larger for corrupted nodes.
  - If the graph is expander, the update will reduce noise.
  - Firing rates are saturated to make this happen.

RESULTS

- The pattern retrieval capacity:

$$S^n / (d_c S)^m$$

  - S: the maximum firing rate.
  - d_c: deg(constraint nodes)

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  - Is exponential if: $\log(S) > m \log(d_c)/(n-m)$

RESULTS

Solution

- The pattern retrieval capacity:

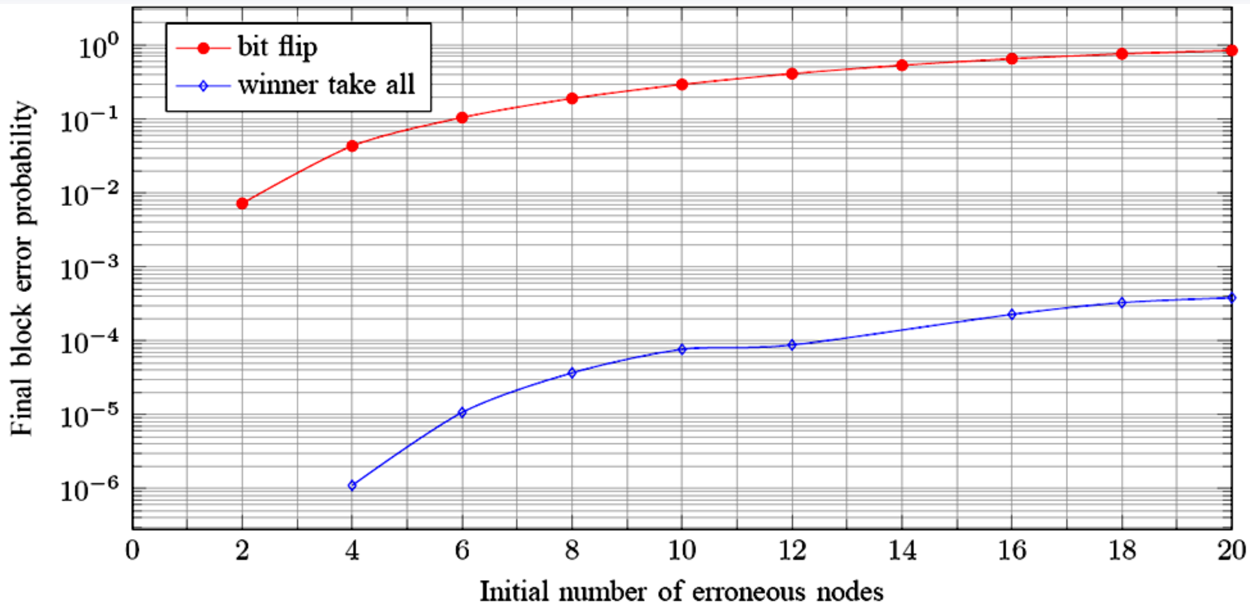
$$S^n/(d_c S)^m$$

  - S: the maximum firing rate.
  - d_c: deg(constraint nodes)
  - Is exponential if: $\log(S) > m\log(d_c)/(n-m)$
- **Theorem**: If H is an expander graph, the *winner-take-all* and *bit-flipping** algorithms are guaranteed to correct two erroneous nodes.

RESULTS (CONTD.)

Solution

- For more than two errors:



- $n = 600$, $m = 300$, $d_c = 10$, $d_p = 5$
- Noise model: integer values in [-5,5]

Ongoing Work

What Happens to H?

- So far, H was assumed to be given.
- In order to find H, there are two possible approaches:

Discussion

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- In order to find H, there are two possible approaches:

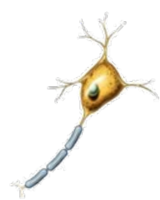
1. Infer H from data

Discussion

WHAT HAPPENS TO H?

- So far, H was assumed to be given.
- In order to find H, there are two possible approaches:

1. Infer H from data

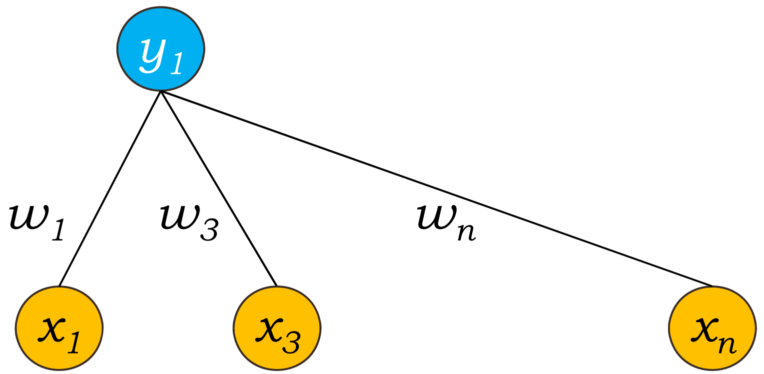


   - Back propagation does not work: H is not guaranteed to be an expander.
   - Low activity assumption might help us here.
   - If data is sparse itself, we might be able to use a similar structure to learn from data

Discussion

Inferring H from Data

- Consider the neighborhood of constraint node one.



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INFERRING H FROM DATA

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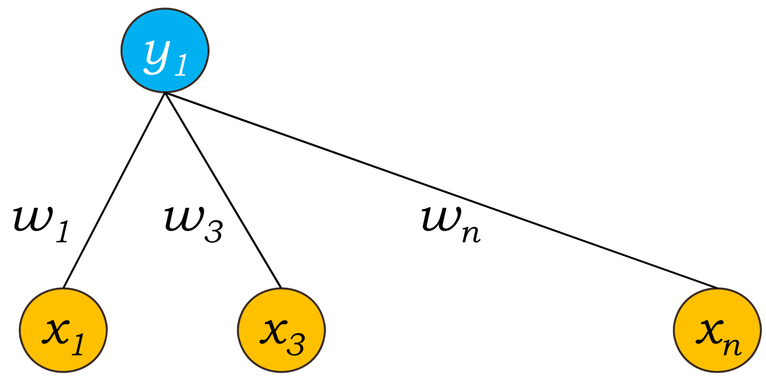
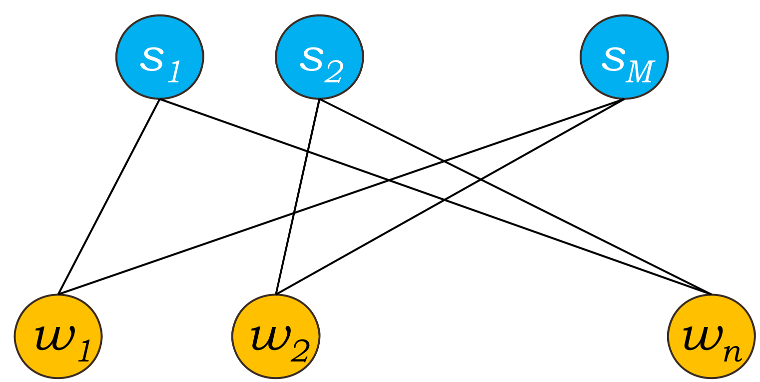


- If the weights are correct, y_1 will be zero <u>for all patterns</u>.
- If not, let of s_μ be the output of y_1 for pattern μ.

Discussion

INFERRING H FROM DATA (CONTD.)

- Consider the following structure:
  - Link weights are patterns.



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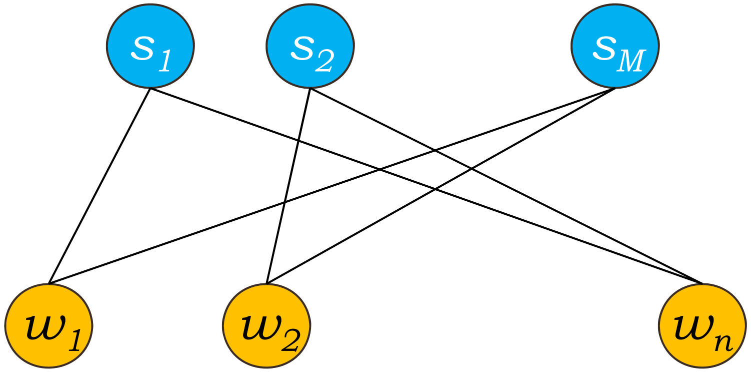
INFERRING H FROM DATA (CONTD.)

- Consider the following structure:
  - Link weights are patterns.
- If $w_1,\ldots, w_n$ are correct, $s_1,\ldots,s_M$ will all be zero.
- If not, we have the same problem as before.



Discussion

Inferring H from Data (contd.)

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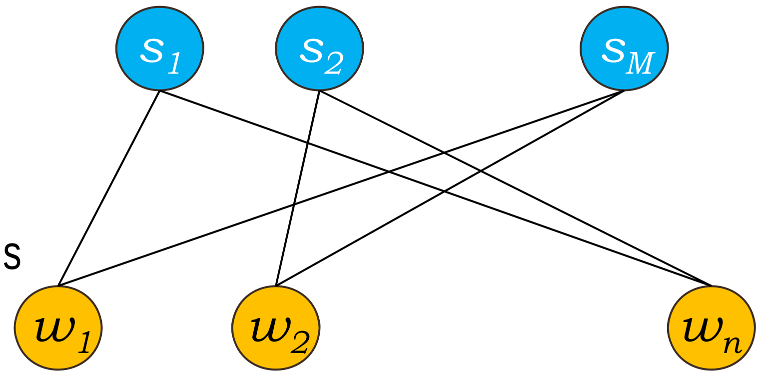
- If $w_1, \ldots, w_n$ are correct, $s_1, \ldots, s_M$ will all be zero.
- If not, we have the same problem as before.

- If we have low activity patterns, this graph is also sparse!
- We might apply the previous algorithm for the learning phase as well!



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WHAT HAPPENS TO H? (CONTD.)

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What Happens to H? (Contd.)

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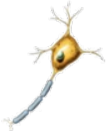

- This corresponds to pre-processing stages in the brain.

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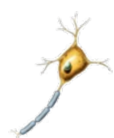
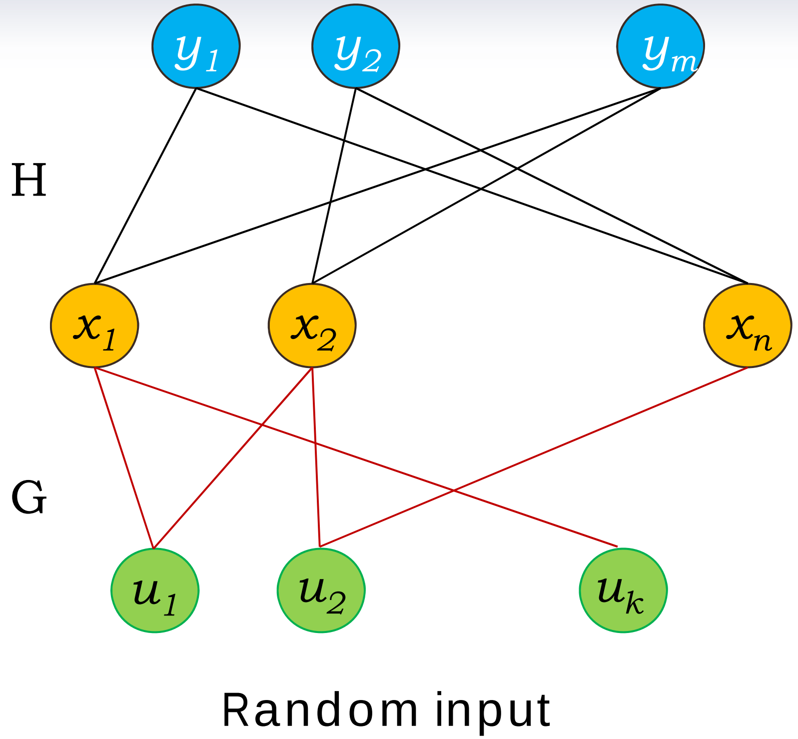


   - This corresponds to pre-processing stages in the brain.

- For the second approach, noise model must be modified.

What Happens to H? (Contd.)

- $u_1,\ldots,u_k$ can be purely random.
- $Gu^\mu = x^\mu$
- $HG = 0$
- G Should be sparse as well.



Discussion

Conclusions and Final Remarks

- An associative memory with exponential capacity is proposed.
  - All that is needed is a two-layer neural network which is also an expander.
  - Simple update rules

Conclusions and Final Remarks

- An associative memory with exponential capacity is proposed.
  - All that is needed is a two-layer neural network which is also an expander.
  - Simple update rules
- We are now working on:
  - Inferring H from data or
  - Mapping input to a higher dimension.

Thanks For Your Attention

Any Questions?