Rank of Random Matrices over Finite Fields

Picture of an instantiation of a random matrix, color coded

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Problem

m, n positive integers

$\mathcal{D}$ probability distribution on $\mathbb{F}_q^{m \times n}$

$X \sim \mathcal{D}$

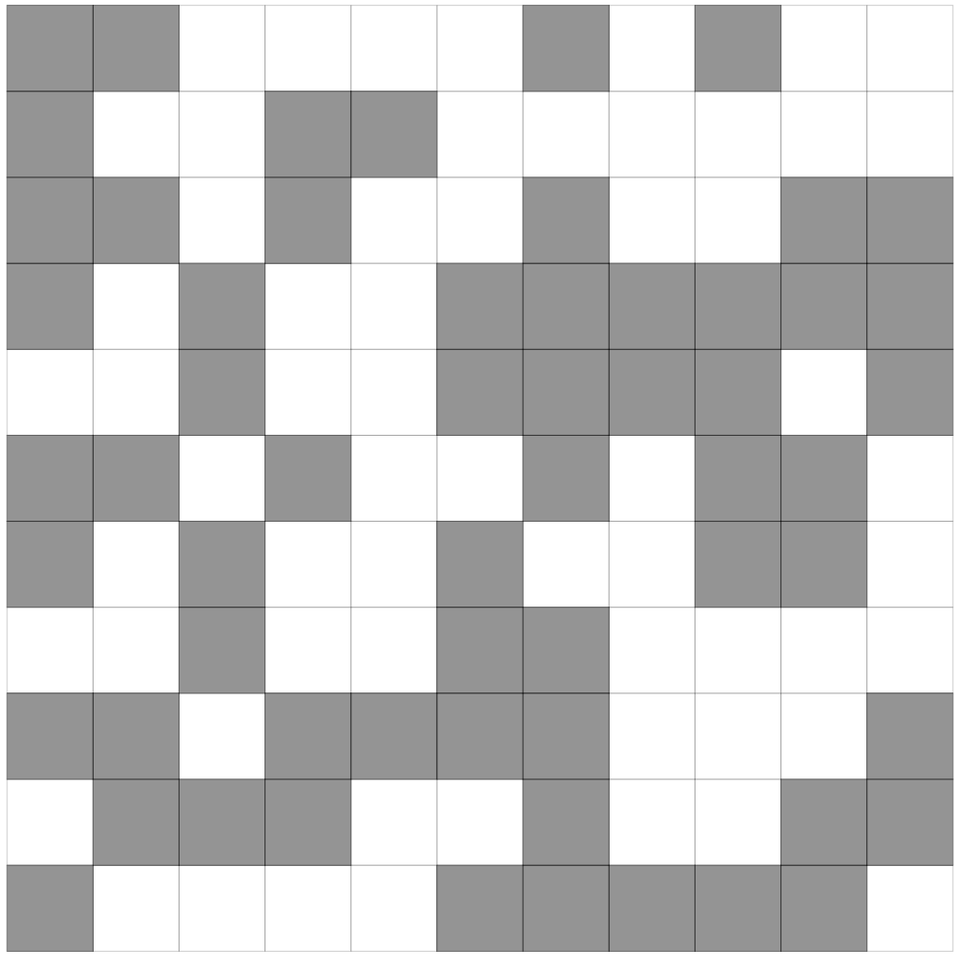
$$\Pr[\mathrm{rk}(X) = m]$$

$$\Pr[\mathrm{rk}(X) = k]$$
$$1 \leq k \leq m$$

Example

$\mathcal{D}$ uniform distribution on $\mathbb{F}_2^{m \times n}$

Every entry is independently chosen to be 1 with prob 0.5



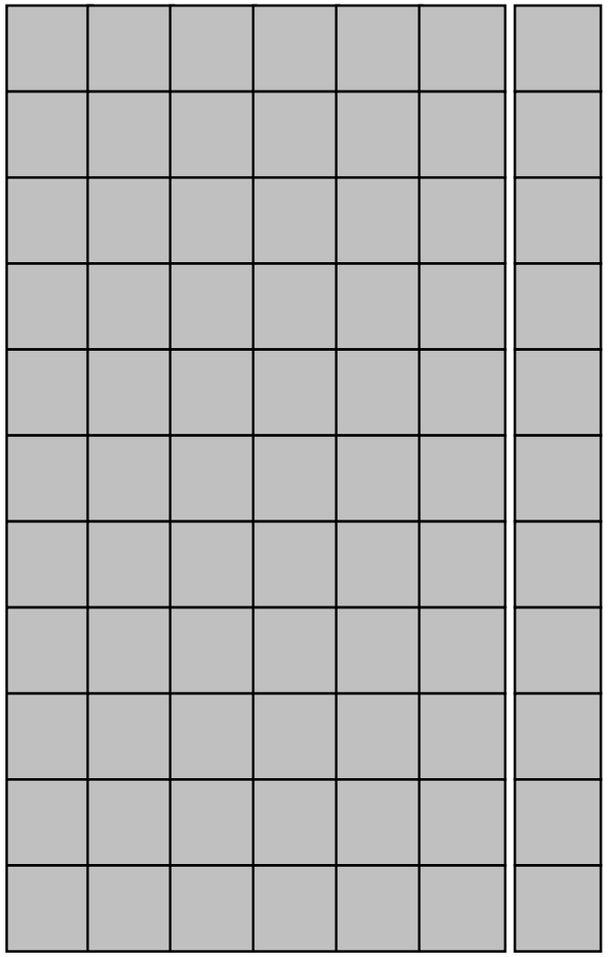
Dynamic Programming

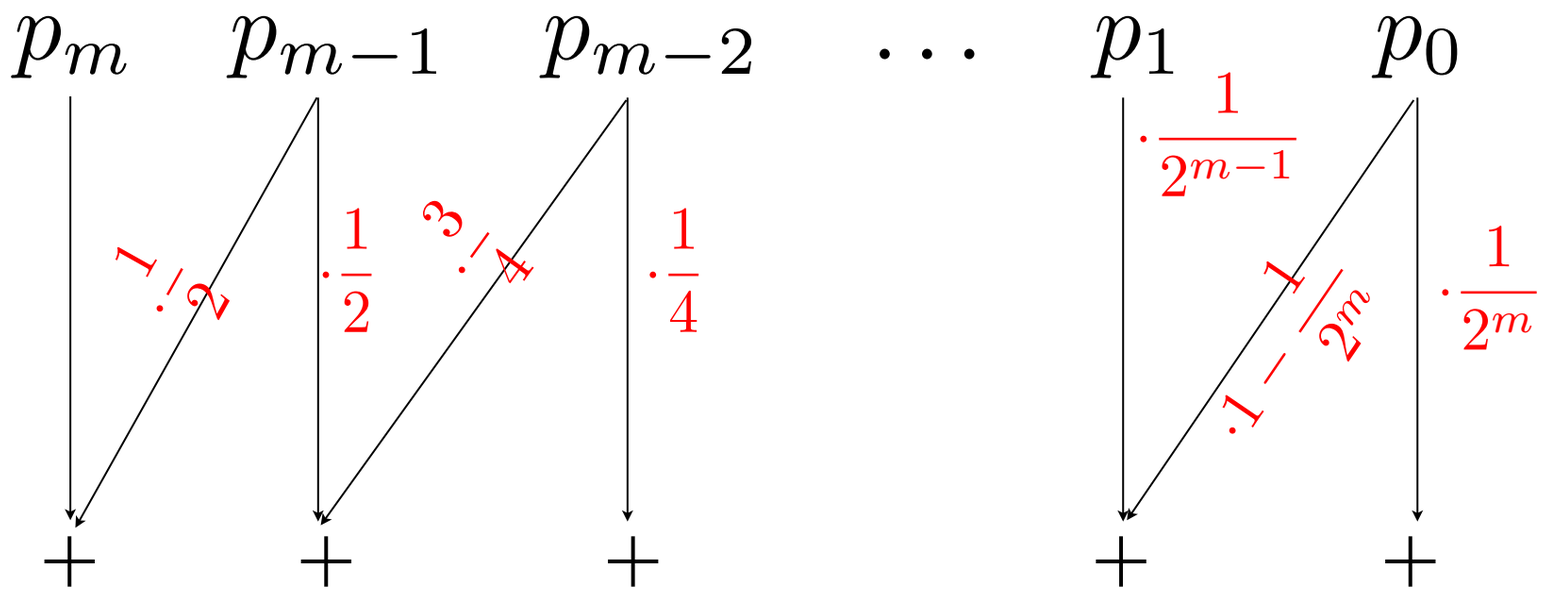
$p_i := \Pr[\mathrm{rk}(X) = i]$

Rank profile: $(p_m, p_{m-1}, \ldots, p_0)$

Add columns one at a time, evolve rank profile

Dynamic Programming





Dynamic Programming

$$\begin{pmatrix} p_m \\ p_{m-1} \\ p_{m-2} \\ \vdots \\ p_1 \\ p_0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{4} & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2^{m-1}} & 1-\frac{1}{2^m} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2^m} \end{pmatrix}^n \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Possible to get approximative analytic expressions

Sparser Matrices

Uniform distribution is not sparse.

Fix $0 \leq p \leq 1$. Each entry is independently 1 with prob p.

Fix $0 \leq p \leq 1$. Each entry is independently nonzero with prob p. If nonzero, then every nonzero element of $\mathbb{F}_q$ appears with equal probability.

Picture goes here

Sparser Matrices

Previous method does not work.

Upper bounds on probability of rank deficiency

Method of Bloemer, Karp, Welzl: Union Bound

$$\begin{aligned}\Pr[\mathrm{rk}(X) < m] &= \Pr[\exists 0 \neq a \in \mathbb{F}_q^m : aX = 0] \\ &\leq \sum_{\mathbb{F}_q^m \ni a \neq 0} \Pr[aX = 0]\end{aligned}$$

Sparser Matrices

Biased distribution: 1 with prob p

Bloemer, Karp, Welzl

Kernel method

Higher moments: matrices over GF(4)

Examples

Other types of randomness:
matrices of fixed row weight
fewer rows than columns

Example: random multigraph
row-weight is 2
Full rank: no cycles
Matrix is cycle-free iff all components are trees
Erdoes-Renyi model of random graphs

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Larger row weights kernel method Klochin's results

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Phase transition: one- or two-sided?

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What about the rank?
Phase transitions?
EXIT charts

What about the rank?
Phase transitions?
EXIT charts

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Larger fields?

GF(q): prob for square matrices

Prob for rectangular matrices?

Applications:

Erasure coding in the random setting

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