A Theory of Coding for Chip-to-Chip Communication

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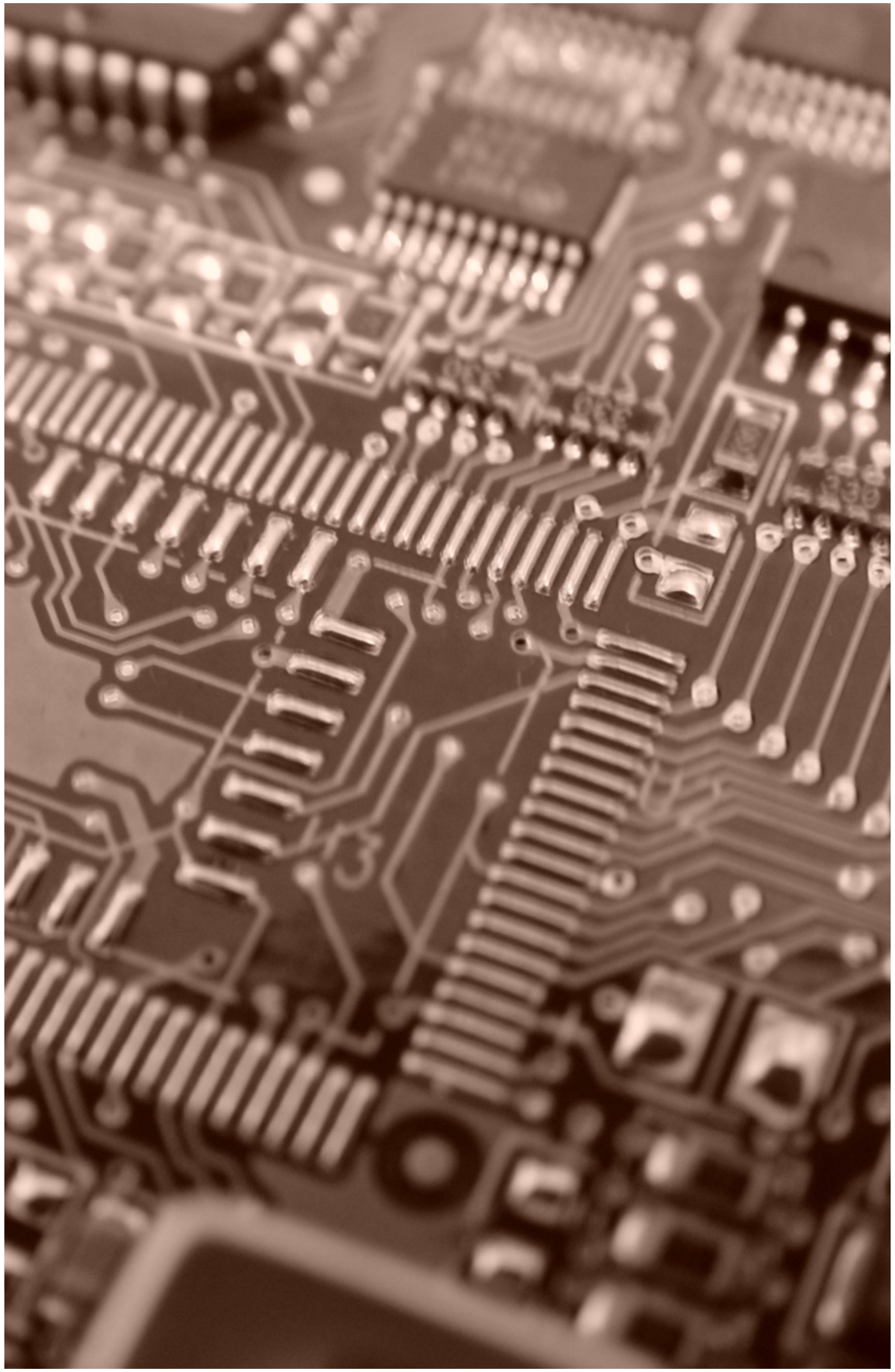
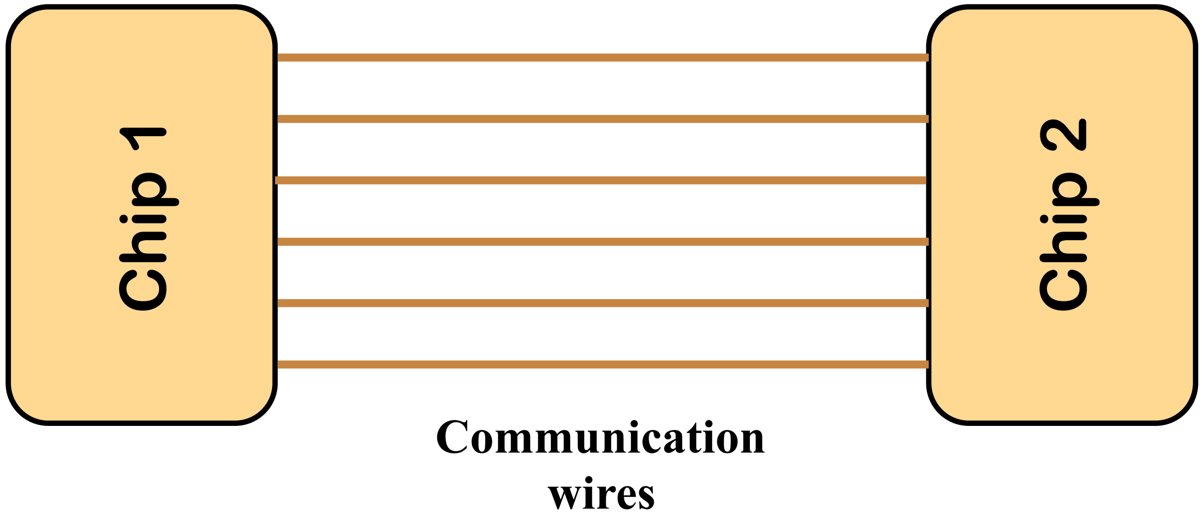
The Problem

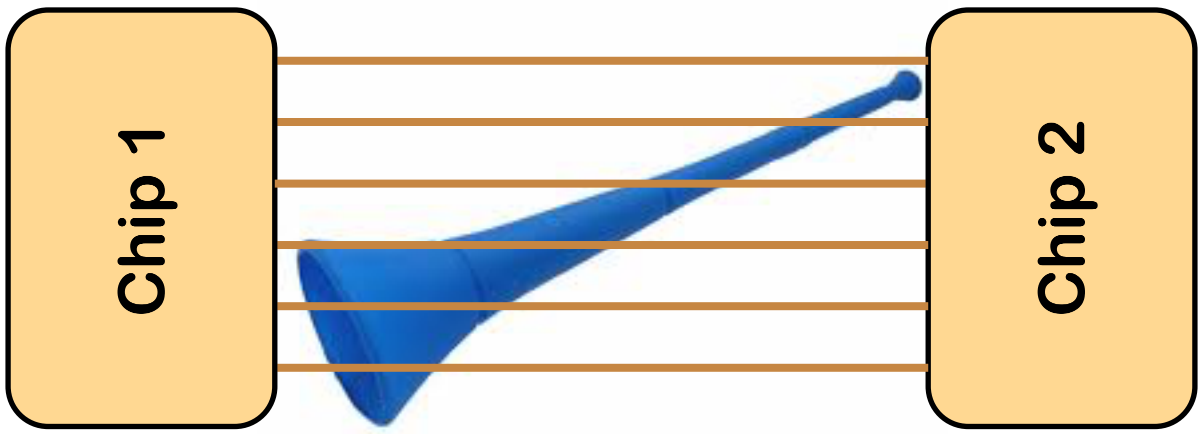


Chip-to-Chip Communication



Abundant....

Noise



Noise scales badly with frequency of transmission:
Example: -40dB at frequency *f*, -90dB at 2*f*

Chordal Codes

Brief intro into theory

Chordal Codes

A (n, N)-chordal code (CC) is

- A finite subset $\mathcal{C} \subset [-1, +1]^n$, $|\mathcal{C}| = N$; Codewords (signals)
- A finite subset $\Lambda \subset \mathbb{R}^n \setminus \{0\}$; Comparators (central hyperplanes)
- And certain constraints. Operational constraints

Parameters

A (n, N)-chordal code (CC) is

- A finite subset $\mathcal{C} \subset [-1, +1]^n$, $|\mathcal{C}| = N$;
- A finite subset $\Lambda \subset \mathbb{R}^n \setminus \{0\}$;
- And certain constraints.

$(\mathcal{C}, \Lambda)$ is (n, N)-CC

- n is called *the number of wires*
- $\log_2(N)/n$ is the *rate* or the *pin-efficiency* #bits per wires
- $|\Lambda|$ is called the *detection complexity*.

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Differential Signaling

$(\mathcal{C}, \Lambda)$ is (n, N)-CC

- n is called *the number of wires*
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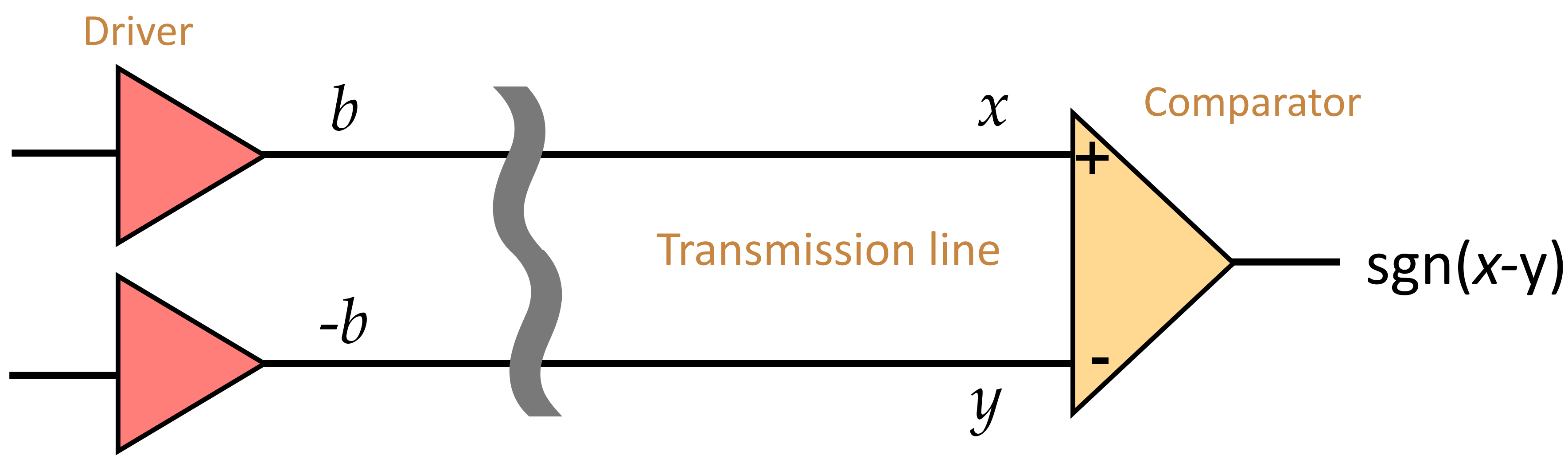
- $|\Lambda|$ is called the *detection complexity*.



Transmits one bit per *a pair* of wires

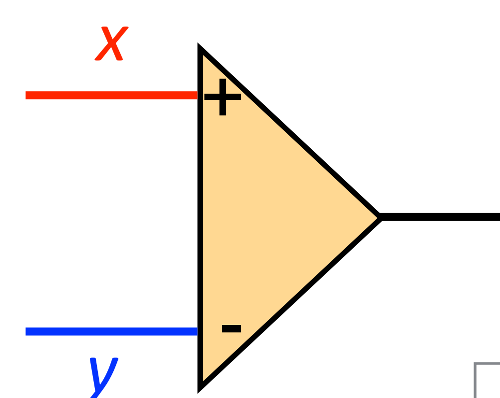
$$\mathcal{C} = \{(1, -1), (-1, 1)\}$$
$$\Lambda = \{(1, -1)\}$$

(2,2)-CC
Rate = 1/2

Electronics: Comparators

Efficient, High-Speed Electronic Circuits

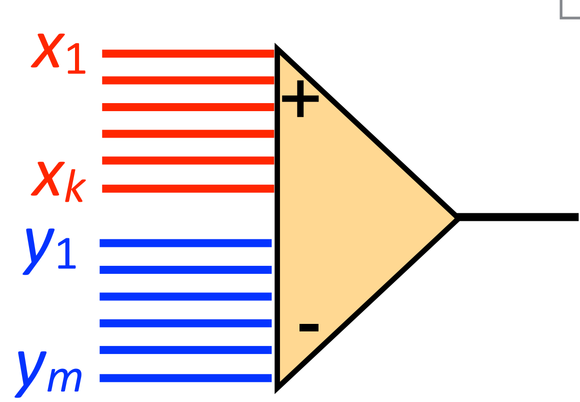
Classical comparators



$\mathrm{sgn}(x - y)$

- Circuit should not have any gain.
- Therefore, only convex combinations allowed.

Multi-Input comparators (MIC)

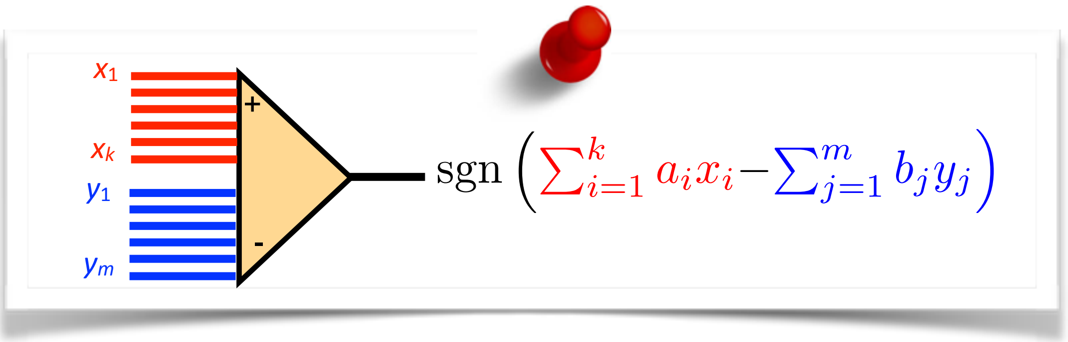


Convex combination

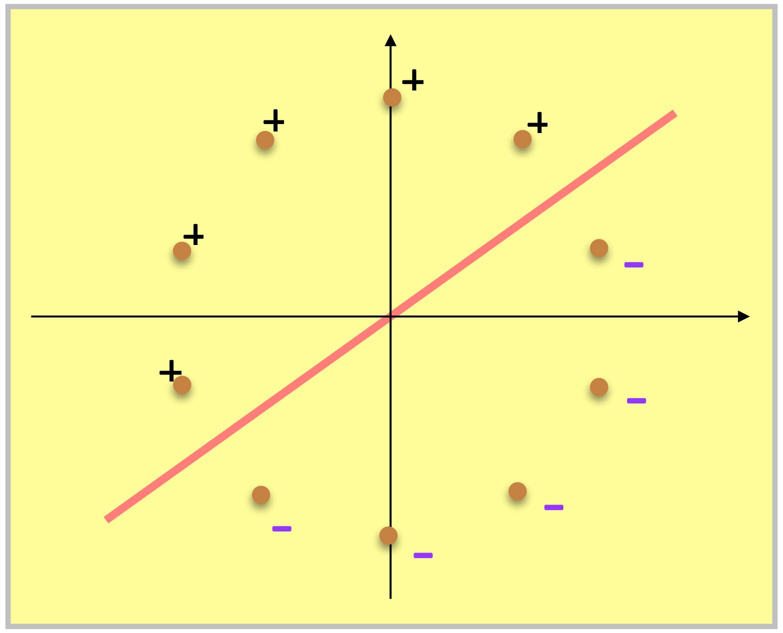
$$\mathrm{sgn}\left(\sum_{i=1}^{k} a_i x_i - \sum_{j=1}^{m} b_j y_j\right)$$

Convex combination

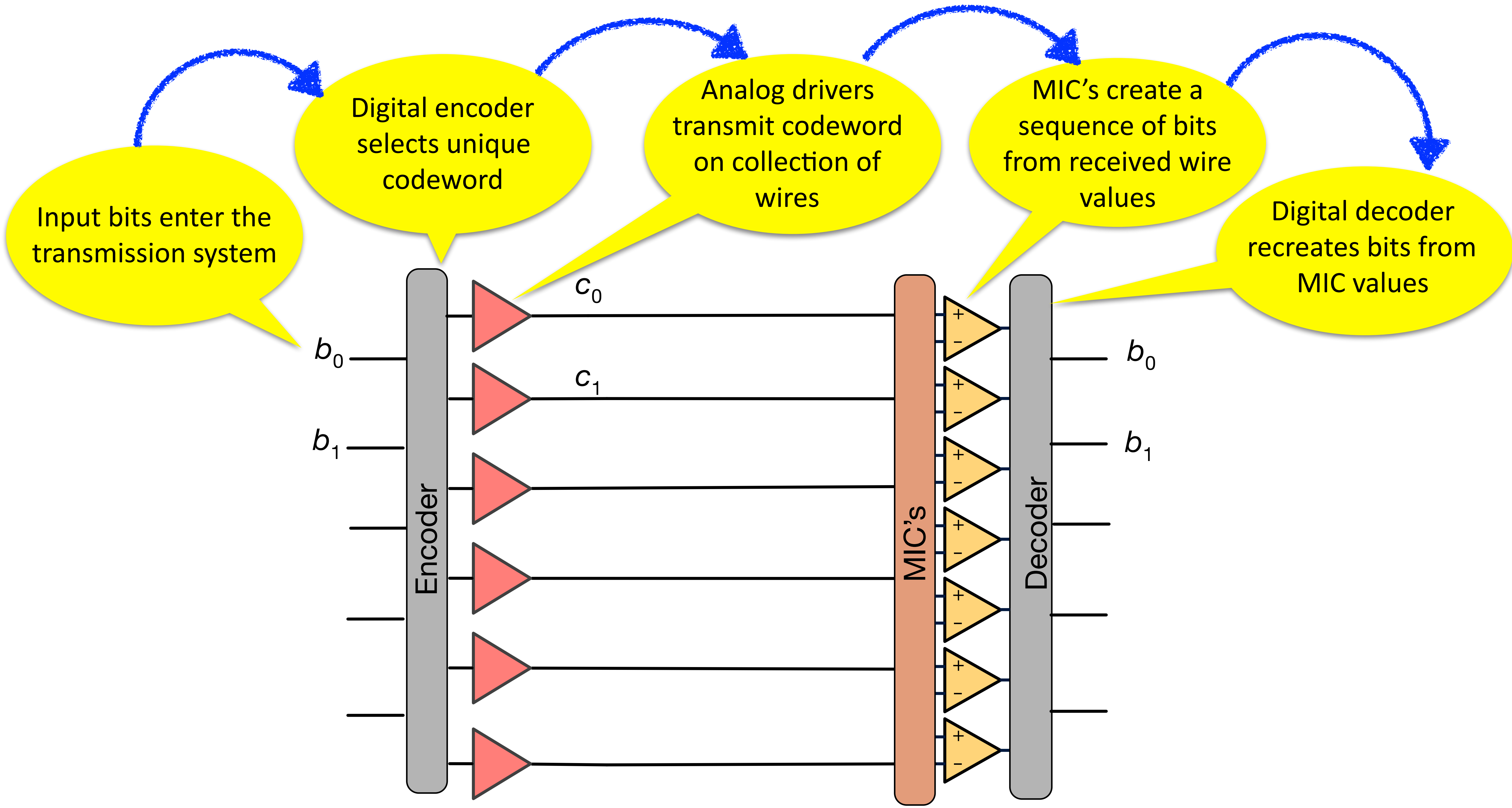
Geometry: Central Hyperplanes



- A MIC corresponds to a central hyperplane
- Each hyperplane subdivides space into two halves
- Each codeword should ideally lie on one side or another
- Not all codewords should lie on the same side



Transmission Chain



- *MIC-signature* of a codeword is sequence of outputs of MIC's.
- Necessary: Every codeword has *unique* MIC signature.

Chordal Codes

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- A finite subset $\Lambda \subset \mathbb{R}^n \setminus \{0\}$;
- And certain constraints.

A (n, N)-chordal code (CC) is

- A finite subset $\mathcal{C} \subset [-1, +1]^n$, $|\mathcal{C}| = N$;
- A finite subset $\Lambda \subset \mathbb{R}^n \setminus \{0\}$;
  - $\forall \lambda \in \Lambda$: $||\lambda||_1 = 2$. No gain
- $\forall c, c' \in \mathcal{C}$, $c \neq c'$, $\exists \lambda \in \Lambda$: $\operatorname{sgn}(\langle \lambda, c \rangle \cdot \langle \lambda, c' \rangle) = -1$.

Unique MIC signature
Distinguishability

First Bound

Given n and $|\Lambda|$, determine the largest N.

What is the largest rate for a given detection complexity?

$$N \leq \sum_{i=0}^{n-1} \binom{|\Lambda|}{i} \left(1 + (-1)^{n-1-i}\right)$$

Zaslavsky's Formula for the max number of chambers of an arrangement of central hyperplanes

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Unbounded Rate

$$N \leq \sum_{i=0}^{n-1} \binom{|\Lambda|}{i} \left(1 + (-1)^{n-1-i}\right)$$

$$|\Lambda| = cn \implies \text{Rate} \sim 1 + \log_2(c)$$

But:

- Asymptotic results are not really relevant
- Didn't take into account noise

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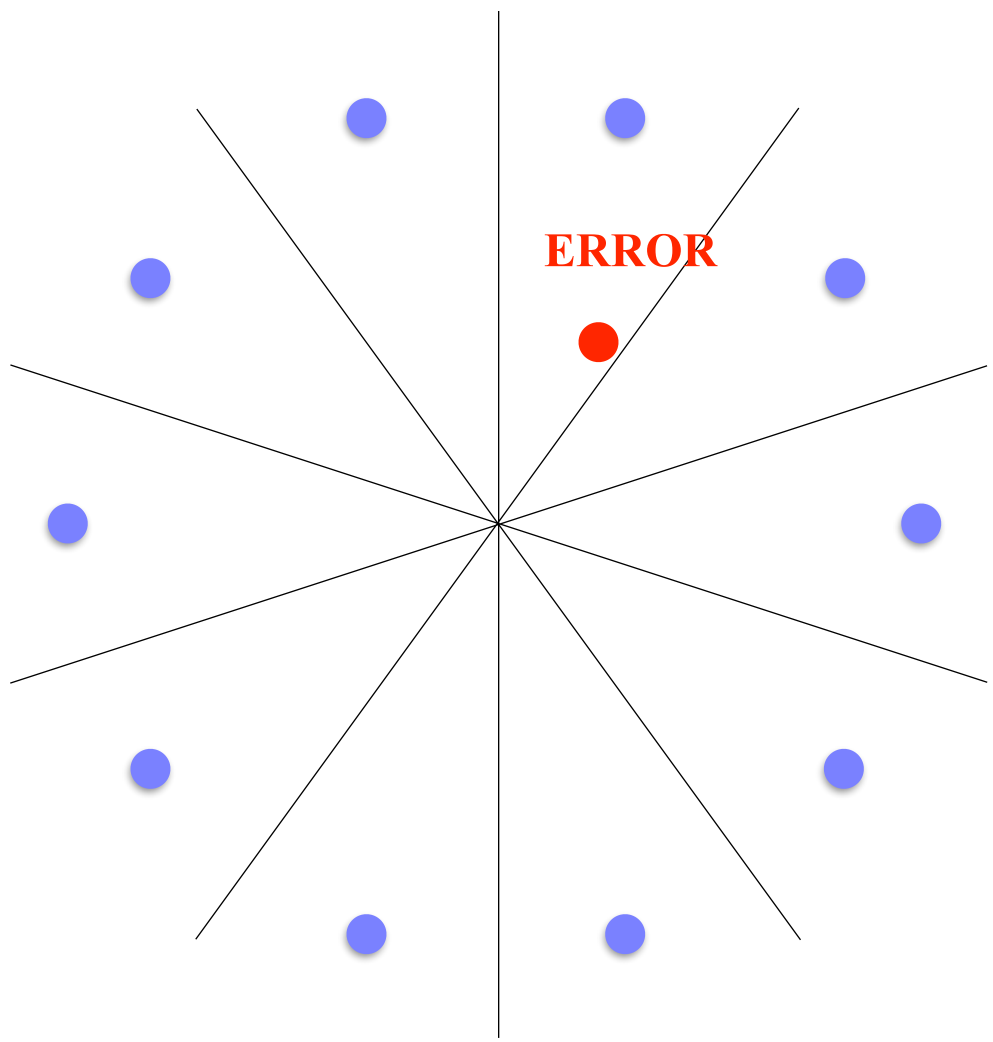
Small Chambers
Susceptibility to Noise



Chordal Codes

A (n, N)-chordal code (CC) is

- A finite subset $\mathcal{C} \subset [-1, +1]^n$, $|\mathcal{C}| = N$;
- A finite subset $\Lambda \subset \mathbb{R}^n \setminus \{0\}$;
  - $\forall \lambda \in \Lambda$: $||\lambda||_1 = 2$.
- $\forall c, c' \in \mathcal{C}$, $c \neq c'$, $\exists \lambda \in \Lambda$: $\mathrm{sgn}(\langle \lambda, c \rangle \cdot \langle \lambda, c' \rangle) = -1$.

$$H := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \textstyle\sum_i x_i = 0\}.$$

A (n, N, I)-chordal code (CC) is

- A finite subset $\mathcal{C} \subset [-1, +1]^n \cap H$, $|\mathcal{C}| = N$.
- A finite subset $\Lambda \subset H$, Common mode resilience
  - $\forall \lambda \in \Lambda$: $||\lambda||_1 = 2$.

Such that

- $\forall c, c' \in \mathcal{C}$, $c \neq c'$, $\exists \lambda \in \Lambda$: $\mathrm{sgn}(\langle \lambda, c \rangle \cdot \langle \lambda, c' \rangle) = -1$.
- $\forall \lambda \in \Lambda, c, c' \in \mathcal{C}$: $\dfrac{|\langle \lambda, c \rangle|}{|\langle \lambda, c' \rangle|} \leq I$. ISI resilience

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Parameters

$H := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i = 0\}$.

A (n, N, I)-chordal code (CC) is

- A subset $\mathcal{C} \in [-1, +1]^n \cap H$, $|\mathcal{C}| = N$.
- A subset $\Lambda \in H \cap L_2$, $\forall \lambda \in \Lambda$: $||\lambda||_2 = 2$.

Such that

- $\forall c, c' \in \mathcal{C}$, $c \neq c'$, $\exists \lambda \in \Lambda$: $\mathrm{sgn}(\langle \lambda, c\rangle \cdot \langle \lambda, c'\rangle) = -1$.
- $\forall \lambda \in \Lambda, c, c' \in \mathcal{C}$: $\dfrac{|\langle \lambda, c\rangle|}{|\langle \lambda, c'\rangle|} \leq I$.

$(\mathcal{C}, \Lambda)$ is (n, N, I)-CC.

- n is called *the number of wires*
- $\log_2(N)/n$ is the *rate* or the *pin-efficiency* #bits per wires
- $|\Lambda|$ is called the *detection complexity*. The fewer comparators the better (for power/area)
- I is called the *ISI-ratio* (if equality holds for some λ, c, c').

Small *I* means better resilience to ISI

Fundamental Problem

$H := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_i x_i = 0\}$.

A (n, N, I)-chordal code (CC) is

- A subset $\mathcal{C} \in [-1, +1]^n \cap H$, $|\mathcal{C}| = N$.
- A subset $\Lambda \in H \cap L_2$, $\forall \lambda \in \Lambda$: $||\lambda||_2 = 2$.

Such that

- $\forall c, c' \in \mathcal{C}$, $c \neq c'$, $\exists \lambda \in \Lambda$: $\mathrm{sgn}(\langle \lambda, c\rangle \cdot \langle \lambda, c'\rangle) = -1$.
- $\forall \lambda \in \Lambda, c, c' \in \mathcal{C}$: $\dfrac{|\langle \lambda, c\rangle|}{|\langle \lambda, c'\rangle|} \leq I$.

Given n and N, determine smallest I such that there is a (n, N, I)-CC.

Alternatively

Given n and I, determine largest N such that there is a (n, N, I)-CC.

Examples
Differential Signaling

$H := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_i x_i = 0$

A (n, N, I)-chordal code (CC) is

- A finite subset $\mathcal{C} \subset [-1, +1]^n \cap H$, $|\mathcal{C}| = N$.
- A finite subset $\Lambda \subset H$,
  - $\forall \lambda \in \Lambda$: $||\lambda||_1 = 2$.

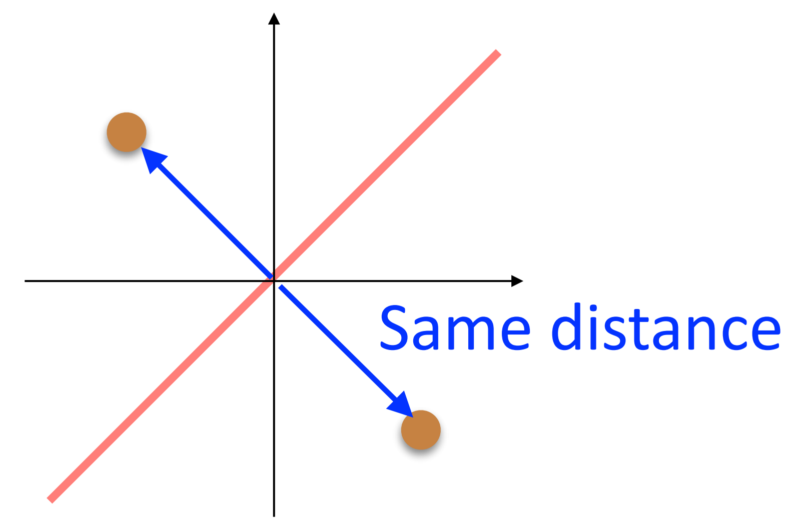
Such that

- $\forall c, c' \in \mathcal{C}$, $c \neq c'$, $\exists \lambda \in \Lambda$: $\mathrm{sgn}(\langle \lambda, c\rangle \cdot \langle \lambda, c'\rangle) = -1$.
- $\forall \lambda \in \Lambda, c, c' \in \mathcal{C}$: $\dfrac{|\langle \lambda, c\rangle|}{|\langle \lambda, c'\rangle|} \leq I$.

$$\mathcal{C} = \{(1, -1), (-1, 1)\}$$
$$\Lambda = \{(1, -1)\}$$

	$(1, -1)$
$(1, -1)$	2
$(-1, 1)$	-2

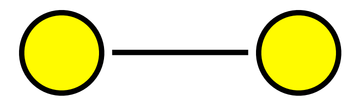
Same magnitude



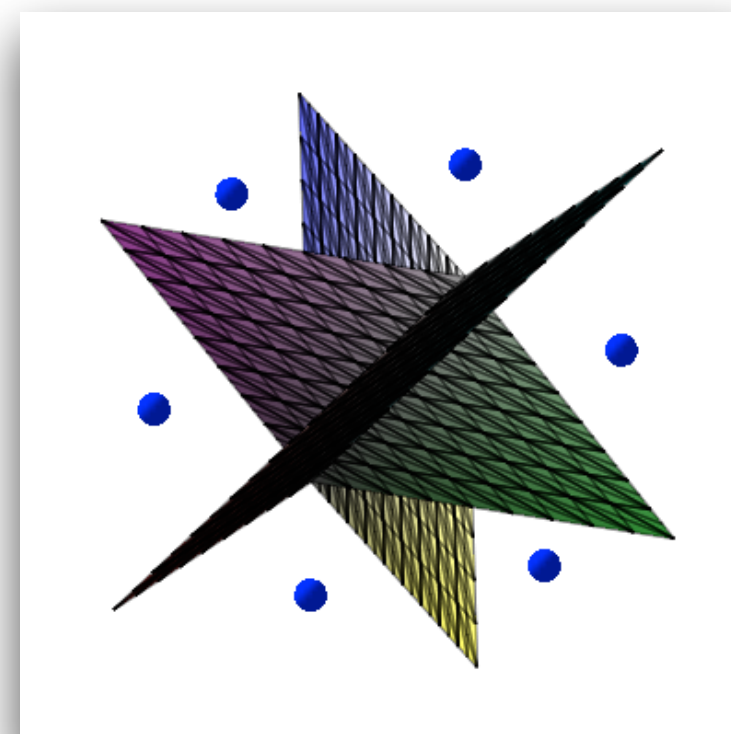
ISIR = 1
(2,2,1)-CC

Examples
3 Wires

$$\mathcal{C} = \{(1,0,-1),(1,-1,0),(0,1,-1),(0,-1,1),(-1,0,1),(-1,1,0)\}$$
$$\Lambda = \{(1,0,-1),(1,-1,0),(0,1,-1)\}$$



Root system A_2



	$(1,0,-1)$	$(1,-1,0)$	$(0,1,-1)$
$(-1,0,1)$	-2	-1	-1
$(-1,1,0)$	-1	-2	1
$(0,-1,1)$	-1	1	-2
$(1,-1,0)$	1	2	-1
$(0,1,-1)$	1	-1	2
$(1,0,-1)$	2	1	1

2x magnitude ratio

ISIR = 2
(3,6,2)-CC

Bounds

$(\mathcal{C}, \Lambda)$ is (n, N, I)-CC.

- $I \geq 1$. Obvious
- $|\Lambda| \geq \log_2(N)$. Every comparator gives at most one bit of information

Constructions

Some, not all....

Tampering Process

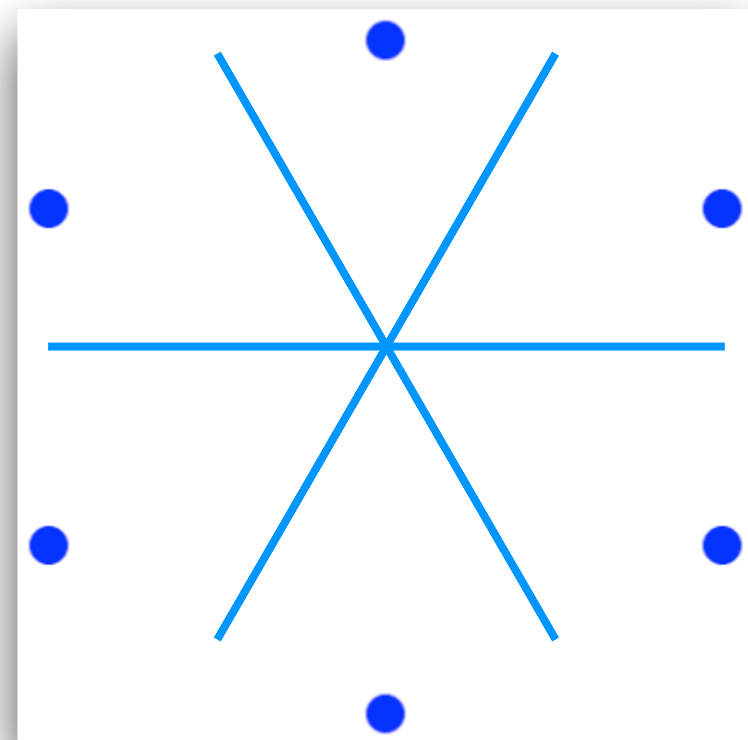
What if sum of coordinates is not zero?

Start with any set of codewords and comparators.

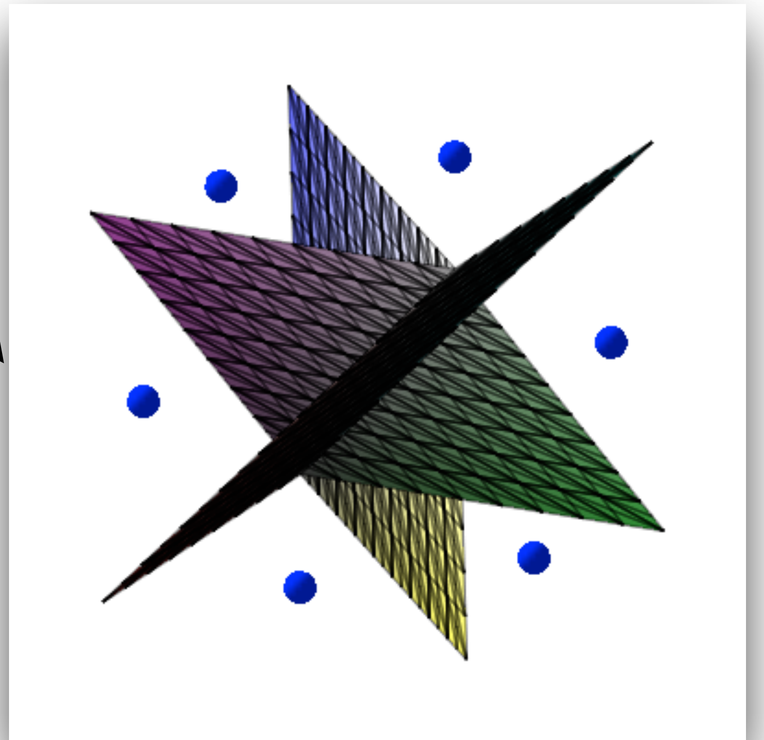
- Construct $(n-1) \times n$-matrix with
  - All rows orthogonal
  - Row-sum $= 0$ for all rows
- $c \in \mathcal{C} : c \cdot A$.
- $\lambda \in \Lambda : \lambda \cdot A$.

Tampering process

Example



$$\begin{pmatrix} 1 & 0 & -1 \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$$



Linear Chordal Codes

Scaling, so coordinates are between ±1

Apply tampering process to

- Vertices of the hypercube and
- The coordinate axes.

$$\mathcal{C} = \frac{1}{m}(\pm 1, \pm 1, \ldots, \pm 1) \cdot A$$

$$\Lambda = \text{scaled versions of rows of } A$$

$$\mathcal{C} = (\pm 1) \cdot (1, -1)$$

$$\Lambda = \{(1, -1)\}$$

Differential

$$\mathcal{C} = \frac{1}{3}(\pm 1, \pm 1, \pm 1) \cdot \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\Lambda = \{(1, -1, 1, -1)/2, (1, 1, -1, -1)/2, (1, -1, -1, 1)/2\}$$

ENRZ

Optimal Chordal Codes

$H := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_i x_i = 0\}$.

A (n, N, I)-chordal code (CC) is

- A finite subset $\mathcal{C} \subset [-1, +1]^n \cap H$, $|\mathcal{C}| = N$.
- A finite subset $\Lambda \subset H$,
  - $\forall \lambda \in \Lambda$: $||\lambda||_1 = 2$.

Such that

- $\forall c, c' \in \mathcal{C}$, $c \neq c'$, $\exists \lambda \in \Lambda$: $\mathrm{sgn}(\langle \lambda, c \rangle \cdot \langle \lambda, c' \rangle) = -1$.
- $\forall \lambda \in \Lambda, c, c' \in \mathcal{C}$: $\frac{|\langle \lambda, c \rangle|}{|\langle \lambda, c' \rangle|} \leq I$.

- For all $n \geq 2$ there exists $(n, 2^{n-1}, 1)$-CC with $n-1$ comparators.
- If $(\mathcal{C}, \Lambda)$ is $(n, N, 1)$-CC, then $N \leq 2^{n-1}$.

- Optimal number of comparators
- Optimal number of codewords
- Maximal rate is asymptotically 1
- Doubles rate of differential signaling

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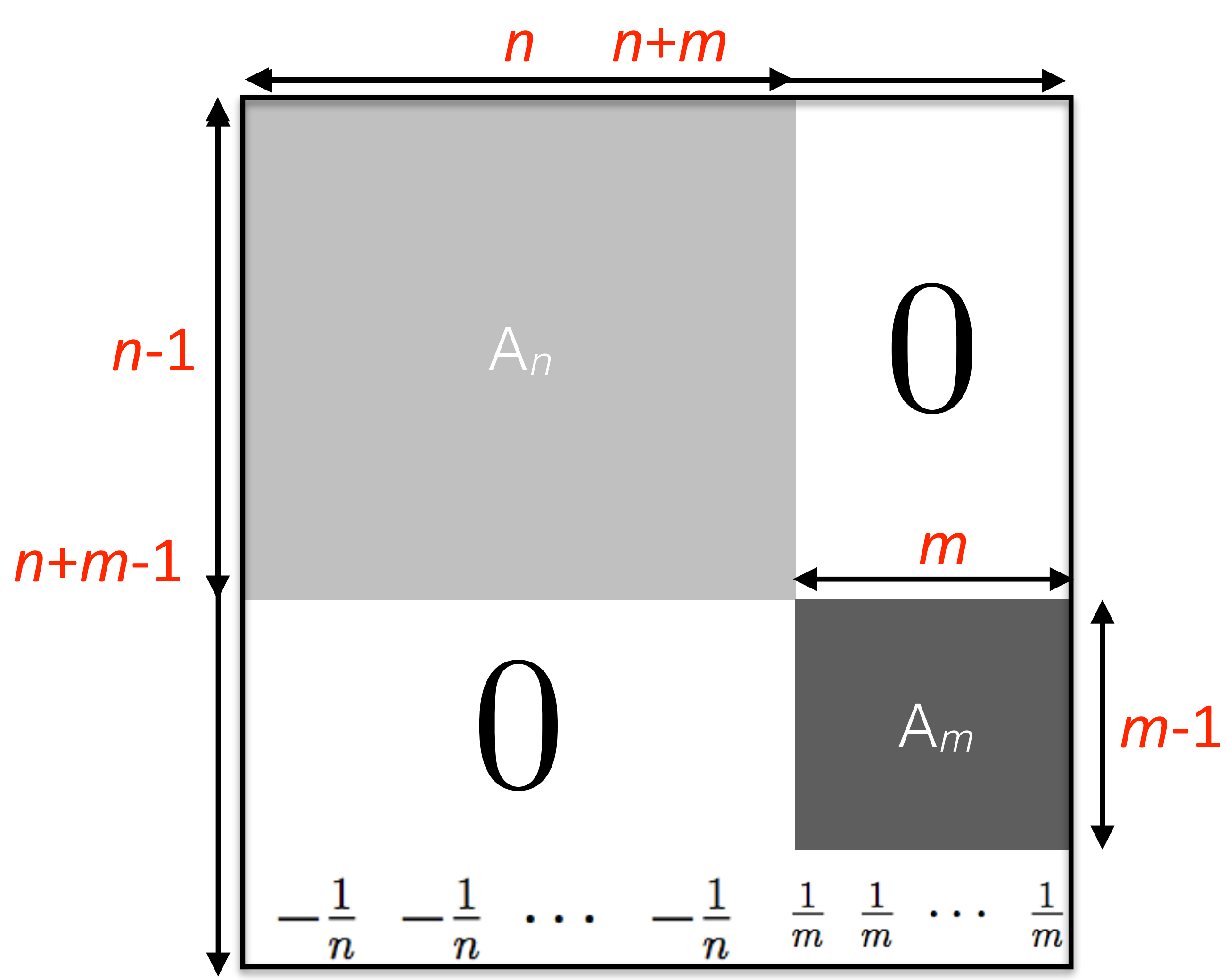
Proofs

- For all $n \geq 2$ there exists $(n, 2^{n-1}, 1)$-CC with $n-1$ comparators.
- If $(\mathcal{C}, \Lambda)$ is $(n, N, 1)$-CC, then $N \leq 2^{n-1}$.

Construct tampering matrix of size n for all $n \geq 2$ by recursion.

$$(n, 2^{n-1}, 1) \wedge (m, 2^{m-1}, 1) \implies (n+m, 2^{n+m-1}, 1).$$



$$A_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 0 & -1 \\ 1/2 & -1 & 1/2 \end{pmatrix}$$

Examples

$$\begin{pmatrix} 1 & -1 & \multicolumn{2}{c}{0} \\ \multicolumn{2}{c}{0} & 1 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Phantom

$$\begin{pmatrix} 1 & 0 & -1 & & 0 & \\ \frac{1}{2} & -1 & \frac{1}{2} & & & \\ & 0 & & 1 & 0 & -1 \\ & & & \frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

CNRZ-5

Other ISI Ratios

- Conjecture: (n, N, I)-CC $\implies N \leq (1+I)^{n-1}$.
- Max rate $\lesssim \log_2(1+I)$
- Can show rate $\sim \log_2(1+I)$ for integer I.

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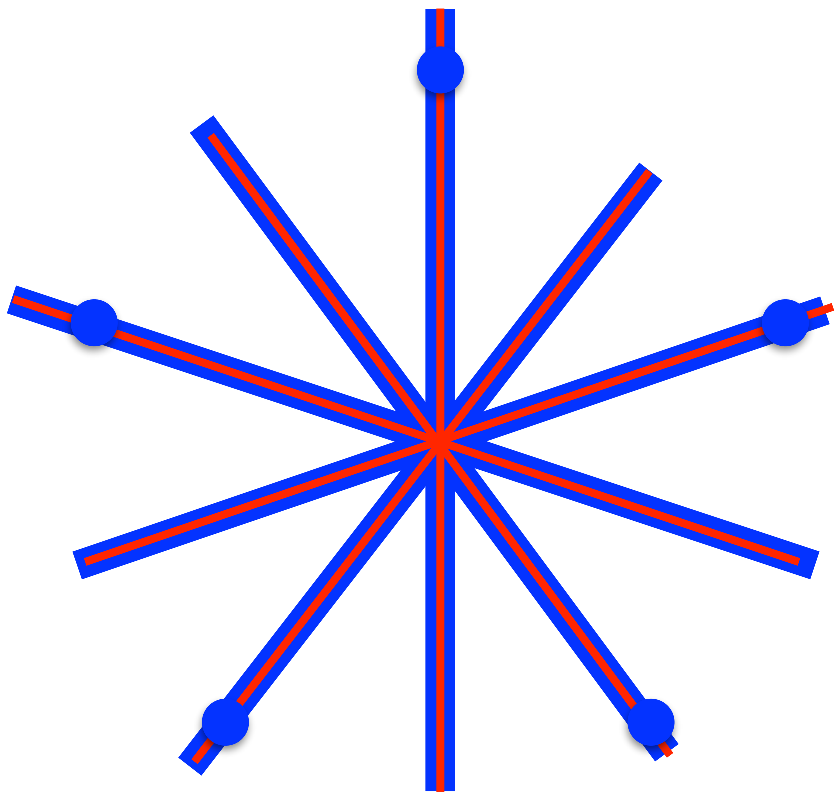
Construction Methods Relaxation

- Define stripe around every hyperplane
  - Codewords inside a stripe are “inactive” for that hyperplane (and vice versa)
  - Codewords outside stripe are “active” for that hyperplane (and vice versa)
- Any two codewords are separated by at least one active hyperplane
- For ISI-ratio only active hyperplanes are considered

Relaxation



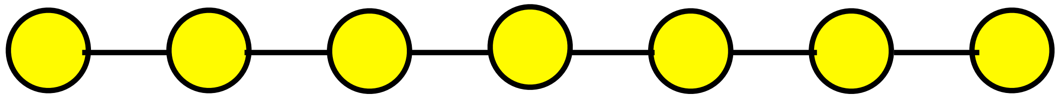
- ISI-ratio without relaxation $= \infty$

Example
Permutation Modulation Codes

- Take a vector $v \in [-1,+1]^n \cap H$.
- Codebook is the orbit of v under S_n (coordinate permutations)
- Comparators are all "pairwise comparators" $e_i - e_j$, $1 \leq i < j \leq n$.

David Slepian

- Rediscovered for chip-to-chip communication by many companies/individuals
- Relaxation: incident codewords and hyperplanes are inactive
- Many comparators….



Root system A_{n-1}

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Example
Maximal Rate

- Fix integer ISI-ratio I.
- Alphabet is equidistant of size $I+1$.
- Vector v has $\sim n/(I+1)$ coordinates equal to any given alphabet element.
- Take PM code generated by v.

$$\text{Rate} = \frac{1}{n}\log_2\binom{n}{\frac{n}{I+1},\frac{n}{I+1},\ldots,\frac{n}{I+1}} = \log_2(1+I) - o(1)$$

Example

What is the best ISI-ratio for $n = 4$, $N = 16$?

Best result so far: 2.38933, 11 comparators
not practical

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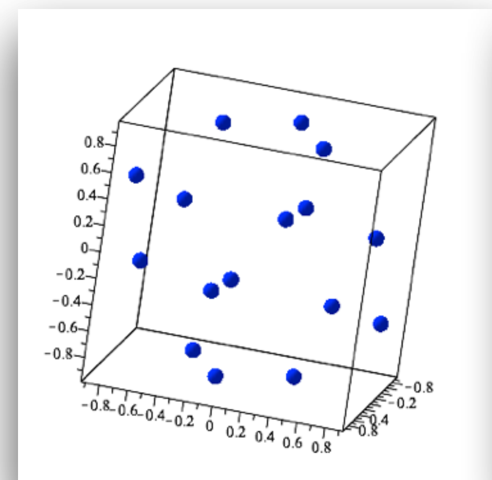
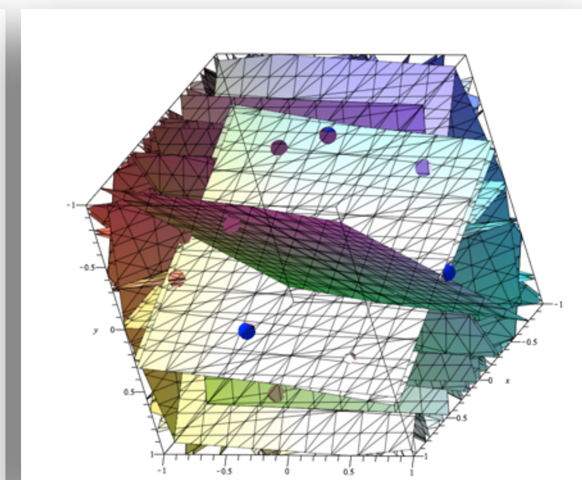
How it was Obtained

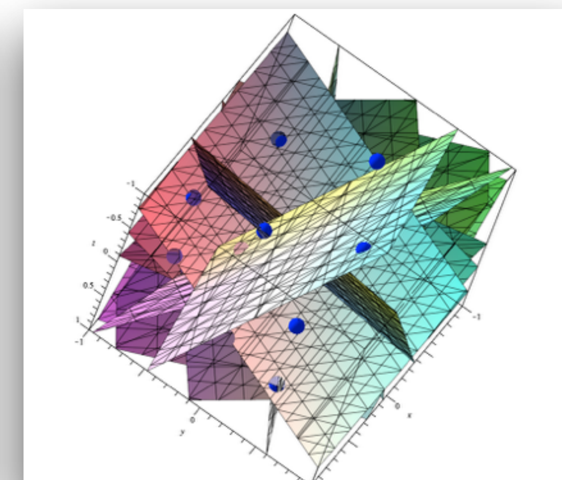
What point set should we start with???



Spherical code of size 16 in three dimensions



Calculate all the bisectors between pairs of points.



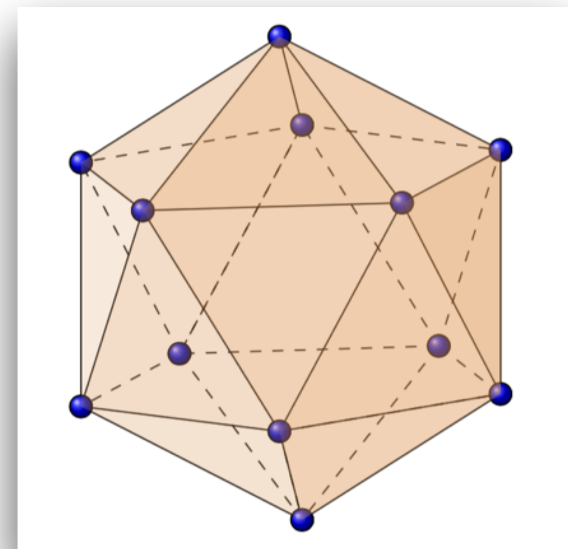
Apply relaxation procedure to points and bisectors to obtain best ISI ratio and smallest number of separating hyperplanes

$$\cdot \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Multiply result with a tampering matrix to project to a chordal code in four dimensions. In this example, the Hadamard matrix is used

Other Examples

Archimedean bodies



$(12, 4, (1+\sqrt{5})/2) - \mathrm{CC}$

15 comparators

Spherical codes

(0.735, 0.404, 0.543)	(0.425, 0.442, -0.789)
(-0.317, 0.470, -0.823)	(0.200, -0.776, -0.597)
(0.052, 0.928, -0.367)	(-0.956, -0.286, -0.055)
(0.707, -0.234, -0.666)	(0.039, -0.162, -0.986)
(0.723, -0.686, -0.075)	(0.068, -0.992, 0.101)
(0.084, -0.644, 0.759)	(0.999, 0.003, -0.025)
(0.738, -0.338, 0.582)	(-0.659, -0.183, -0.729)
(-0.468, -0.158, 0.869)	(0.717, 0.680, -0.147)
(-0.497, -0.797, -0.340)	(-0.560, -0.726, 0.397)
(-0.489, 0.860, 0.139)	(-0.864, 0.281, 0.417)
(0.238, 0.0521, 0.969)	(-0.858, 0.402, -0.316)
(0.220, 0.907, 0.357)	(-0.277, 0.555, 0.780)

$(24, 4, 2.69) - \mathrm{CC}$

48 comparators

Permutation modulation codes of type II

$(1, \sqrt{2}-1, \sqrt{2}-1)$	$(-1, \sqrt{2}-1, \sqrt{2}-1)$
$(1, -\sqrt{2}-1, \sqrt{2}-1)$	$(-1, -\sqrt{2}-1, \sqrt{2}-1)$
$(1, \sqrt{2}-1, -\sqrt{2}-1)$	$(-1, \sqrt{2}-1, -\sqrt{2}-1)$
$(1, -\sqrt{2}-1, -\sqrt{2}-1)$	$(-1, -\sqrt{2}-1, -\sqrt{2}-1)$
$(\sqrt{2}-1, 1, \sqrt{2}-1)$	$(-\sqrt{2}-1, 1, \sqrt{2}-1)$
$(\sqrt{2}-1, -1, \sqrt{2}-1)$	$(-\sqrt{2}-1, -1, \sqrt{2}-1)$
$(\sqrt{2}-1, 1, -\sqrt{2}-1)$	$(-\sqrt{2}-1, 1, -\sqrt{2}-1)$
$(\sqrt{2}-1, -1, -\sqrt{2}-1)$	$(-\sqrt{2}-1, -1, -\sqrt{2}-1)$
$(\sqrt{2}-1, \sqrt{2}-1, 1)$	$(-\sqrt{2}-1, \sqrt{2}-1, 1)$
$(\sqrt{2}-1, -\sqrt{2}-1, 1)$	$(-\sqrt{2}-1, -\sqrt{2}-1, 1)$
$(\sqrt{2}-1, \sqrt{2}-1, -1)$	$(-\sqrt{2}-1, \sqrt{2}-1, -1)$
$(\sqrt{2}-1, -\sqrt{2}-1, -1)$	$(-\sqrt{2}-1, -\sqrt{2}-1, -1)$

$(24, 4, \sqrt{2}+1) - \mathrm{CC}$

9 comparators

Subset of Root system B_n

State of Affairs

Exact code values are widely unknown except for $n = 2$.

- Even for case of ISI-ratio 1 under relaxation
  - Does there exist a $(n, > 2^{n-1}, 1)$-CC under relaxation?
- Good idea about the case $n = 3$, but otherwise...

Applications

Maybe some other time….