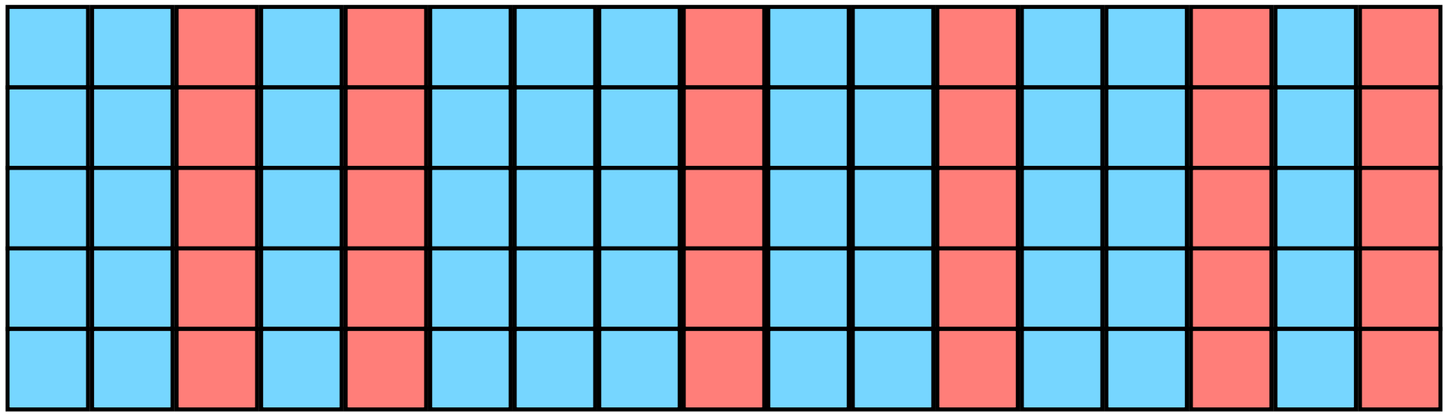
(Improved) Decoding of Interleaved RS-Codes



Amin Shokrollahi

Joint work with A. Brown and L. Minder

EPFL

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Synopsis

1. RS-Codes

2. The Welch-Berlekamp Decoder

3. RS-Codes over Large Alphabets

4. Generalized WB-Decoding: the BKY-algorithm

5. New Analysis

6. Algorithmic Issues

7. AG-Codes

8. Final Remarks

RS-Codes

$x_1, \ldots, x_n \in \mathbb{F}_q$ pairwise distinct

$\mathbb{F}_q[x]_{<k}$ space of polynomials of degree <k, $k \leq n$

$$\begin{array}{rcl} \varphi\colon \mathbb{F}_q[x]_{<k} & \longrightarrow & \mathbb{F}_q^n \\ f & \longmapsto & (f(x_1), \ldots, f(x_n)) \end{array}$$

$\mathrm{Im}(\varphi)$ is called a RS-code $\mathcal{C}$

A nonzero polynomial of degree <k over a field has at most k-1 roots over the field.

$$\ker(\varphi) = \{f \mid f(x_1) = \cdots = f(x_n) = 0\} = 0.$$

$$0 \neq f \in \mathbb{F}_q[x]_{<k} \Rightarrow \#\{i \mid f(x_i) = 0\} < k \Rightarrow \text{Minimum distance of } \mathcal{C} \geq n-k+1.$$

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$$\mathcal{C} \text{ is } [n, k, n-k+1]_q - \text{code}$$

Unique Decoding Problem

$$\left(\quad f(x_1) \quad , \ldots , \quad f(x_n) \quad \right)$$

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Unique Decoding Problem

$$\begin{pmatrix} f(x_1) & , \ldots , & f(x_n) \end{pmatrix}$$

Transmission

$$\begin{pmatrix} y_1 & , \ldots , & y_n \end{pmatrix}$$

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Unique Decoding Problem

$$\underset{\text{Known}}{(\quad y_1 \quad ,\ldots,\quad y_n \quad)} \xleftarrow{\text{Transmission}} \underset{\text{Unknown}}{(\quad f(x_1) \quad ,\ldots,\quad f(x_n) \quad)}$$

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Error positions $E := \{i \mid y_i \neq f(x_i)\}$

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Promise: $\#E =: e \leq \dfrac{n-k}{2}$

Unique Decoding Problem

$$\Big(\quad f(x_1) \quad , \ldots , \quad f(x_n) \quad \Big) \text{ Unknown}$$

Transmission

$$\text{Known } \Big(\quad y_1 \quad , \ldots , \quad y_n \quad \Big)$$

Error positions $E := \{i \mid y_i \neq f(x_i)\}$

Promise: $\#E =: e \leq \dfrac{n-k}{2}$

Decoding problem: Find f.

The Error Locator

Error positions $E := \{i \mid y_i \neq f(x_i)\}$

Error locator: $h(x) := \prod_{i \in E} (x - x_i)$

Zeros of error locator yield the error locations

Finding the error locator reveals *f*:

We know the values of *f* at the *n*-*e* correct positions. Since *n*-*e* > *k*-1, interpolation yields *f*.

How do we find the error locator?

$$\forall\, i\colon h(x_i)(f(x_i) - y_i) = 0.$$

The Welch-Berlekamp Decoder

$$h(x)f(x) - yh(x)|_{(x_i,y_i)} = 0$$

$h(x) \in \mathbb{F}_q[x]_{<\frac{n+k}{2}}$ $\quad$ $yh(x) \in \mathbb{F}_q[x]_{\leq\frac{n-k}{2}}$

Theorem: Any pair $(g, h) \in \mathbb{F}_q[x]_{<\frac{n+k}{2}} \times \mathbb{F}_q[x]_{\leq\frac{n-k}{2}} \setminus \{(0, 0)\}$ such that

$$g(x) - yh(x)|_{(x_i,y_i)} = 0$$

for i=1,...,n, satisfies $f = \frac{g}{h}$

First Proof

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Assume that the zero codeword was sent, and that 1,...,e are the error positions.

$$\begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} e \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} n-e \geq \frac{n+k}{2} \end{array} \left(\begin{array}{cccc|cccc} 1 & x_1 & \cdots & x_1^{d-1} & y_1 & y_1x_1 & \cdots & y_1x_1^{\ell} \\ 1 & x_2 & \cdots & x_2^{d-1} & y_2 & y_2x_2 & \cdots & y_2x_2^{\ell} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_e & \cdots & x_e^{d-1} & y_e & y_ex_e & \cdots & y_ex_e^{\ell} \\ \hline 1 & x_{e+1} & \cdots & x_{e+1}^{d-1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{d-1} & 0 & 0 & \cdots & 0 \end{array}\right) \cdot \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{d-1} \\ \hline h_0 \\ h_1 \\ \vdots \\ h_{\ell} \end{pmatrix} = 0 \qquad \begin{array}{l} \ell = \frac{n-k}{2} \\ d = \frac{n+k}{2} \end{array}$$

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\end{pmatrix}
\cdot
\begin{pmatrix}
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= 0$$

(e rows; $n - e \geq \frac{n+k}{2}$ rows)

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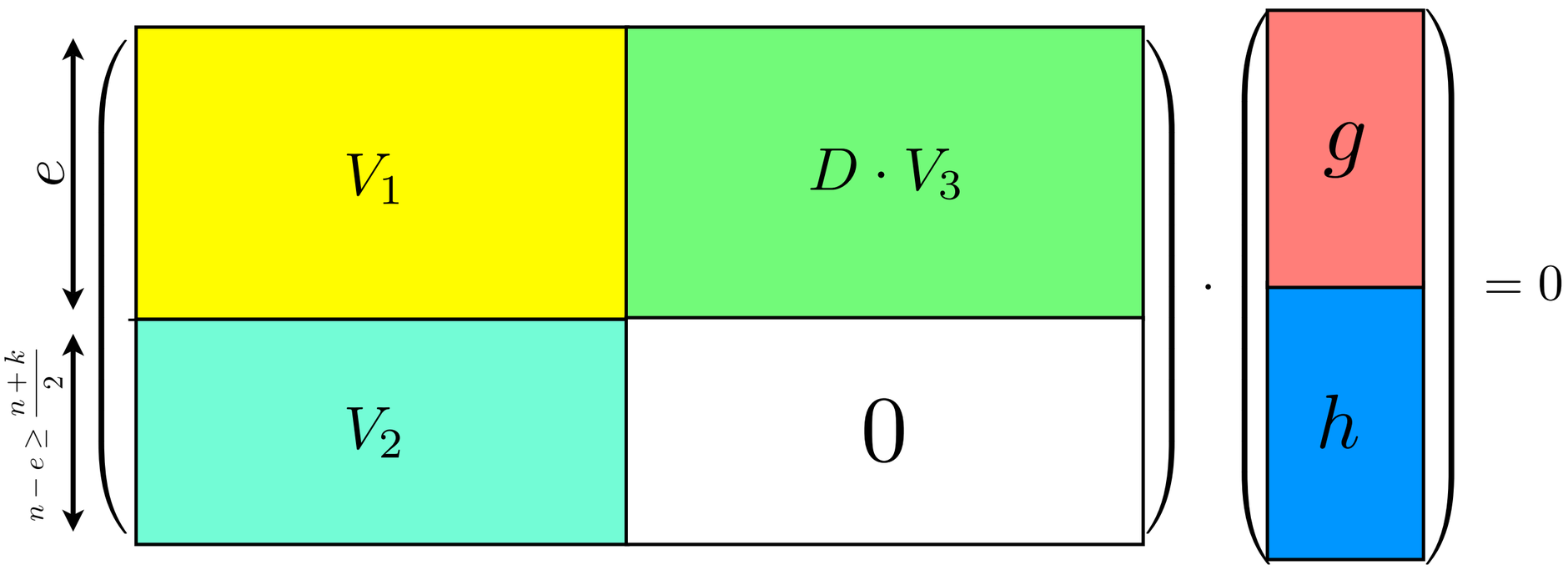

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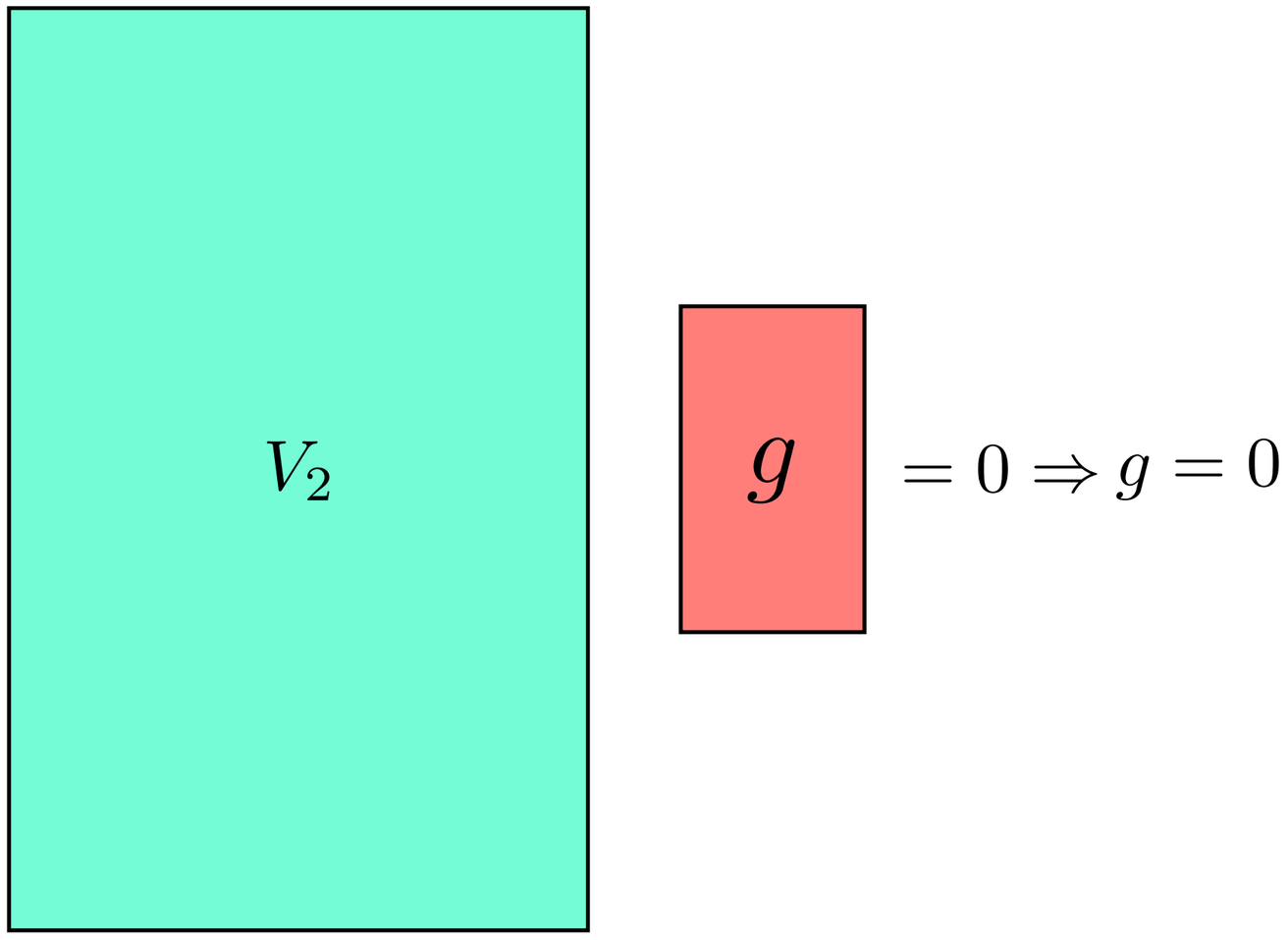


V_2 is Vandermonde, so columns are independent.

Decoding More Errors?

Need other techniques to decode more errors.

If number of errors is larger than $(n\text{-}k)/2$, then the error vector is not necessarily unique.

We could do either (a) list-decoding, or (b) probabilistic decoding.

How many errors can we possible expect to decode?

No more than $(n\text{-}k)$

Decoding More Errors?

List-decoding algorithms of Sudan, and Guruswami-Sudan can decode up to a

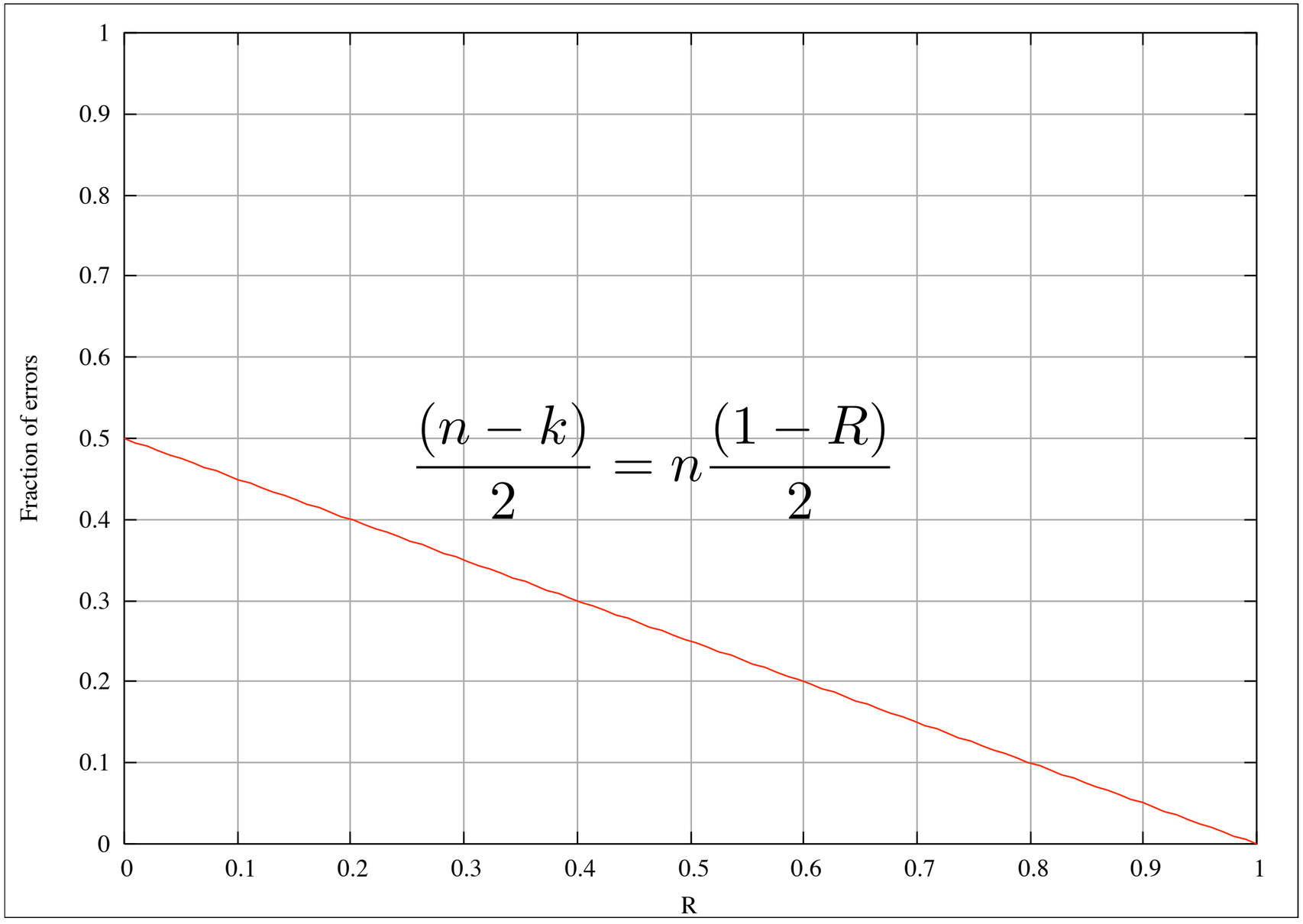
$$1 - \sqrt{k/n}$$

fraction of errors, and produce a short list of possible codewords.

What about more errors?

Possible, under a probabilistic model. Algorithm is due to Bleichenbacher, Kiyaias, and Yung (BKY).

Decoding More Errors?



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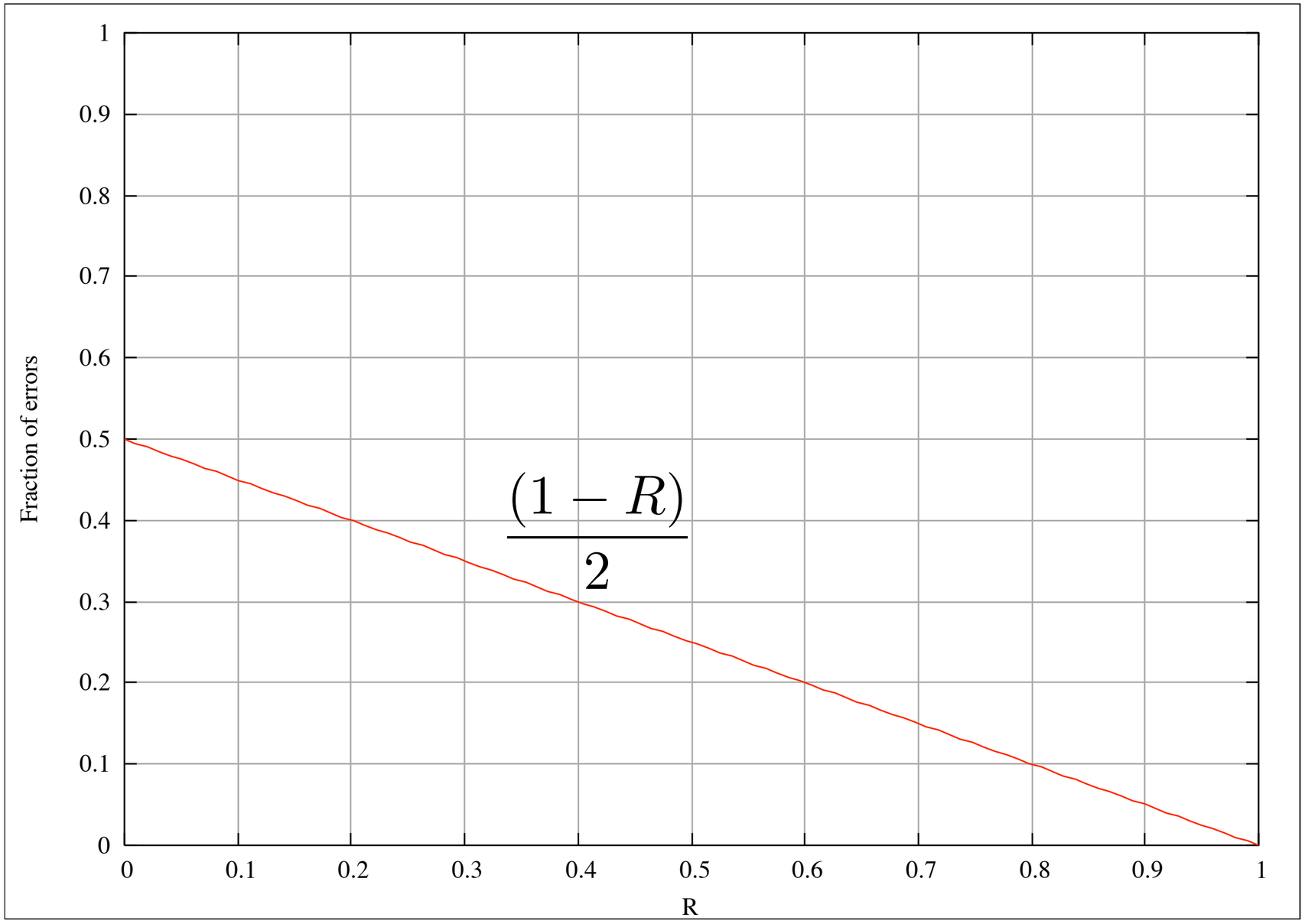

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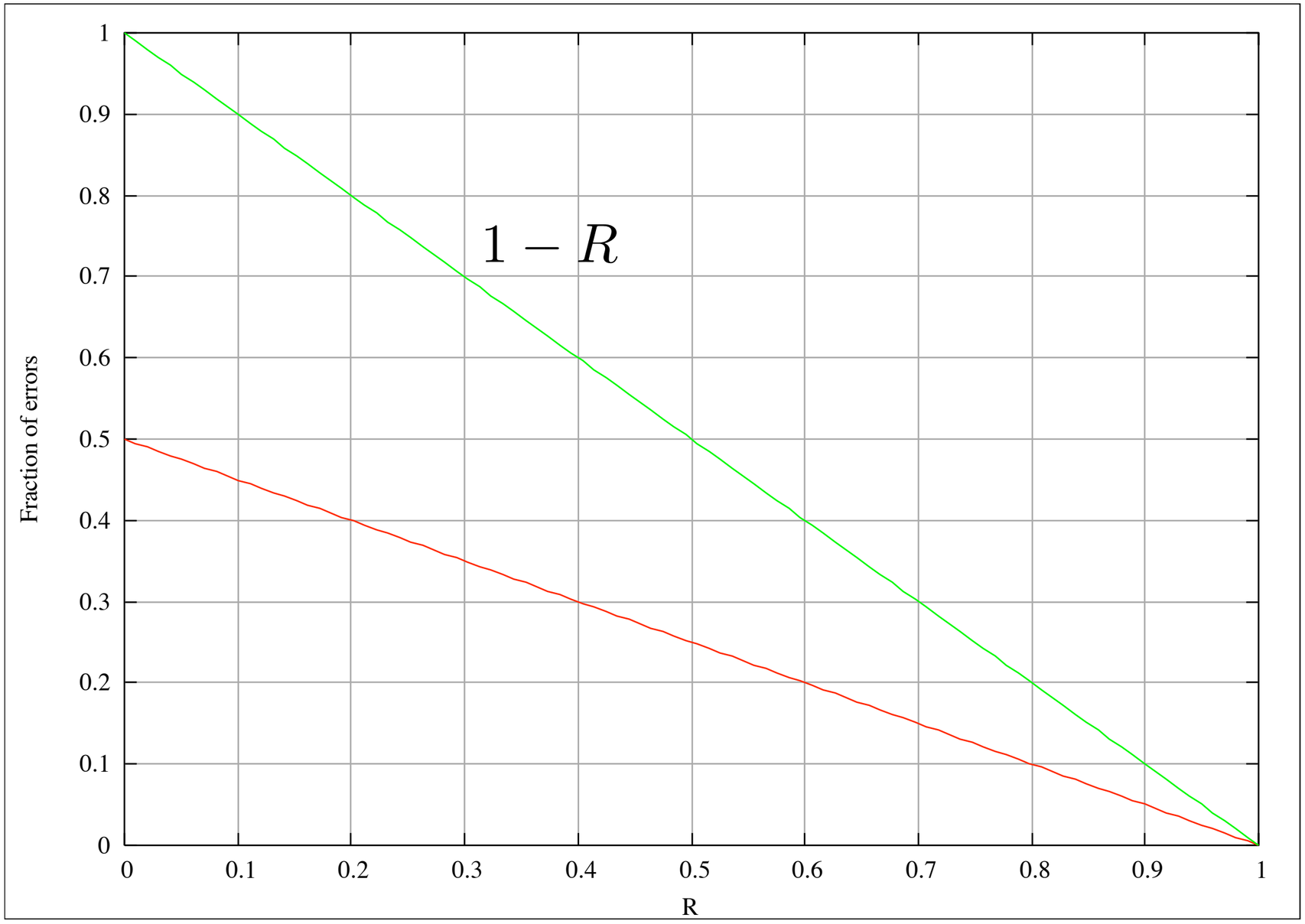

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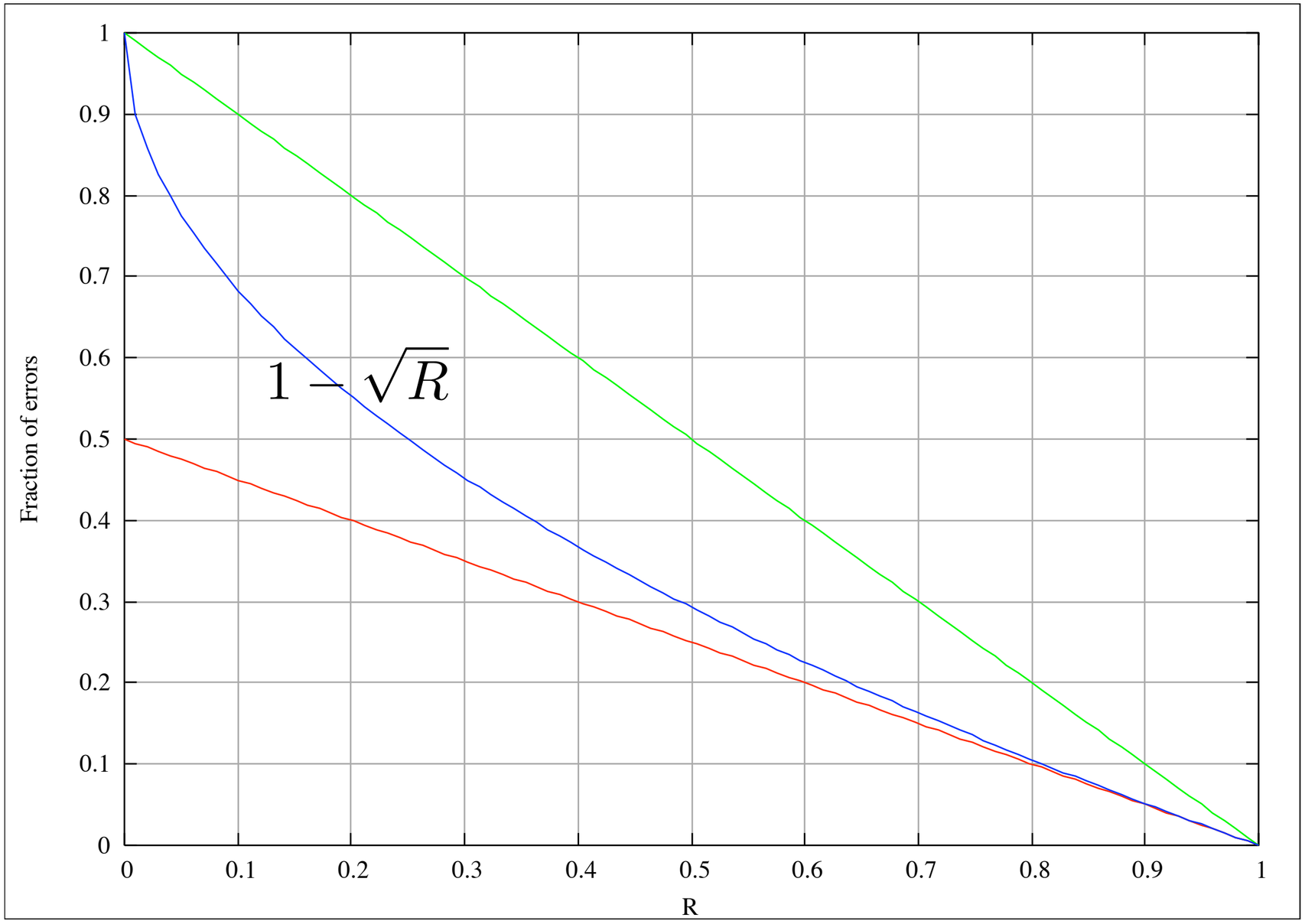

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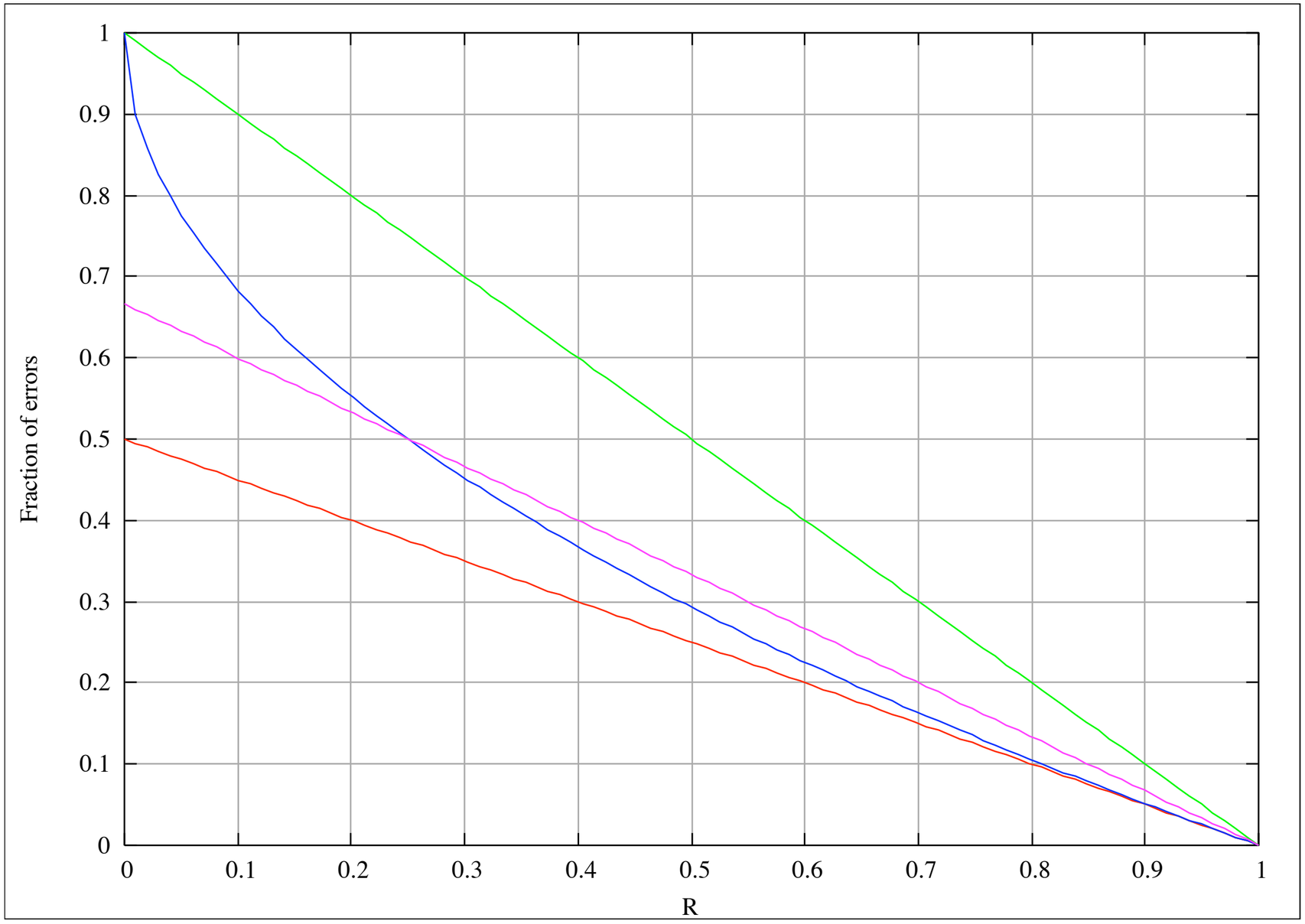

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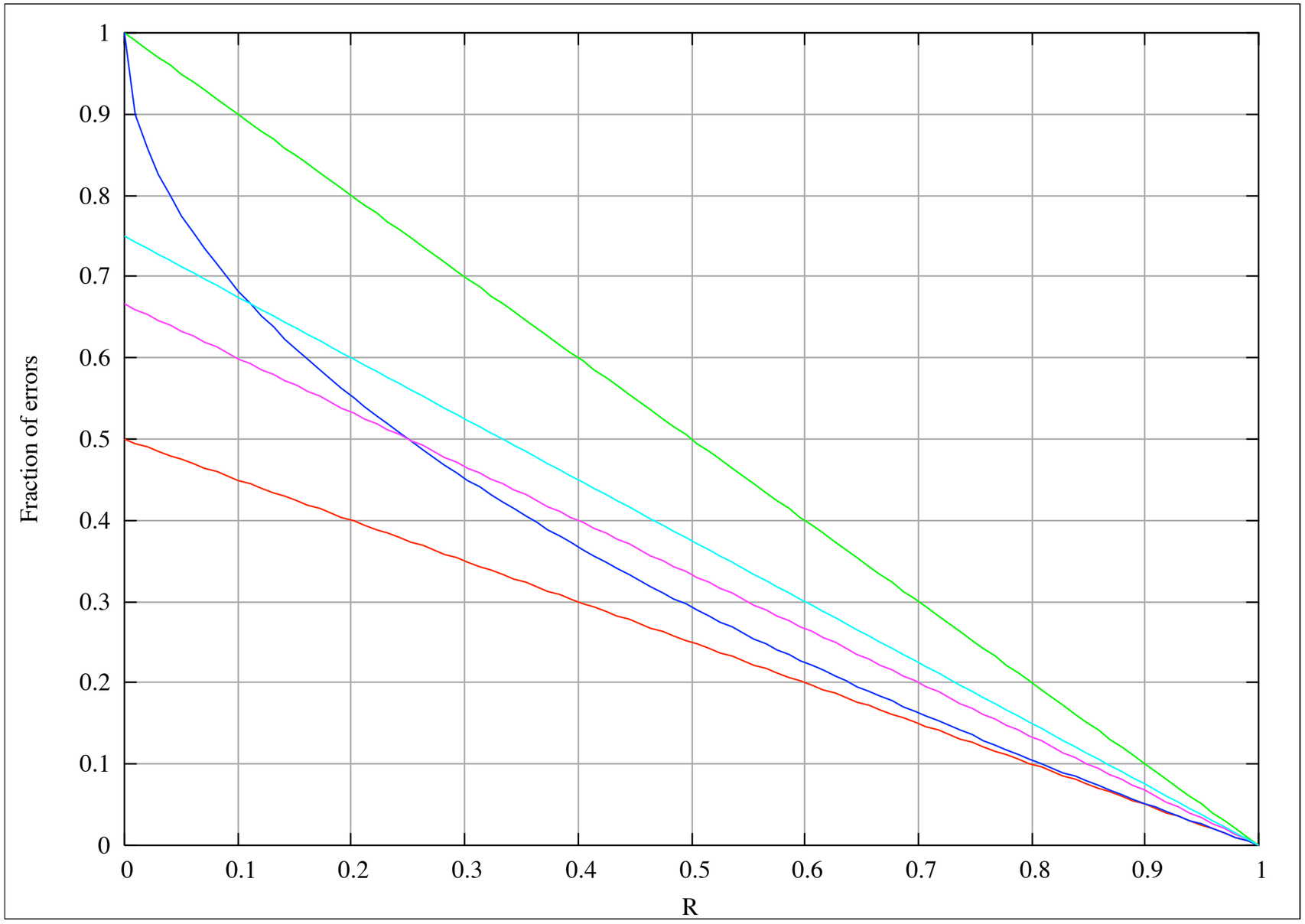

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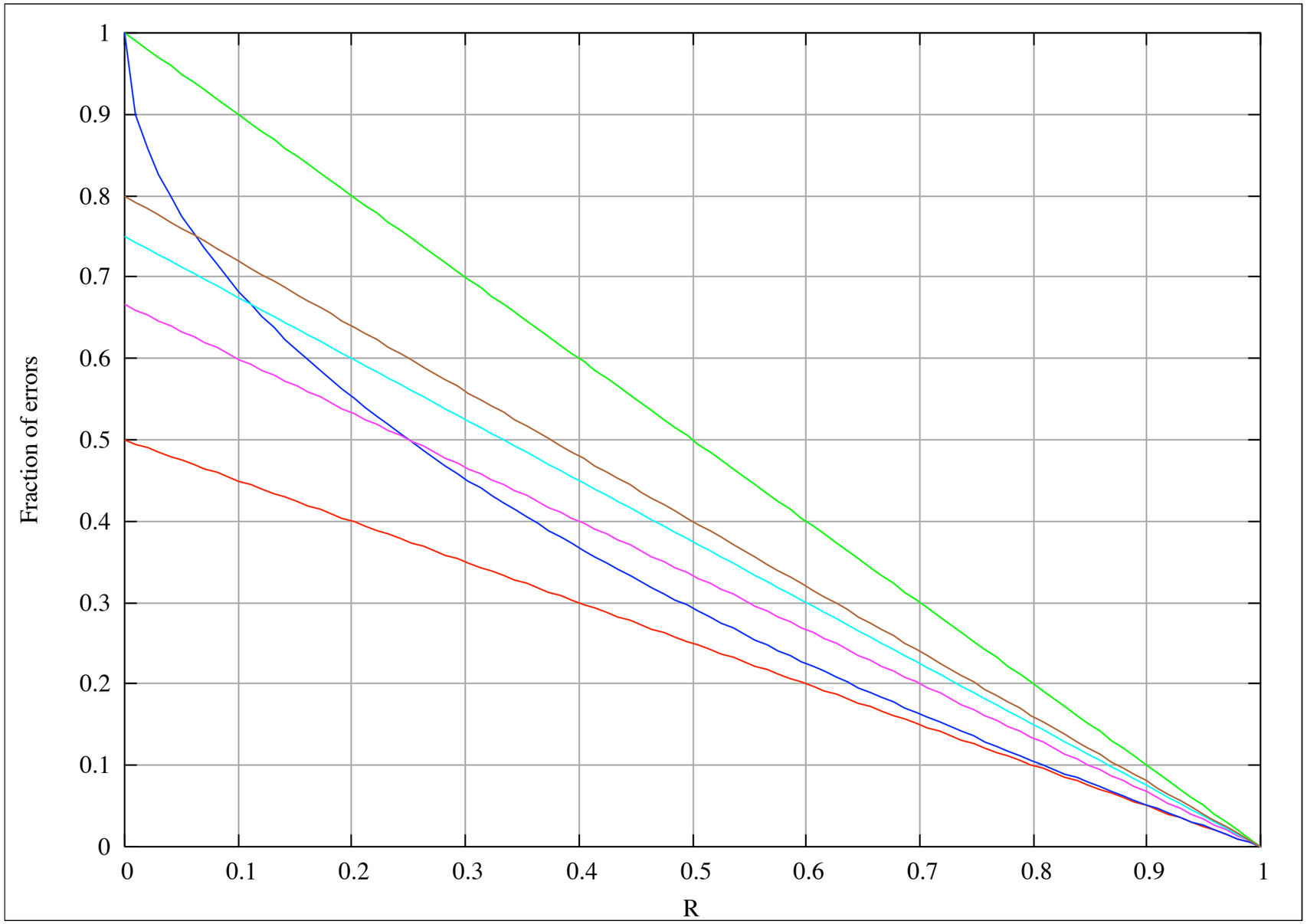

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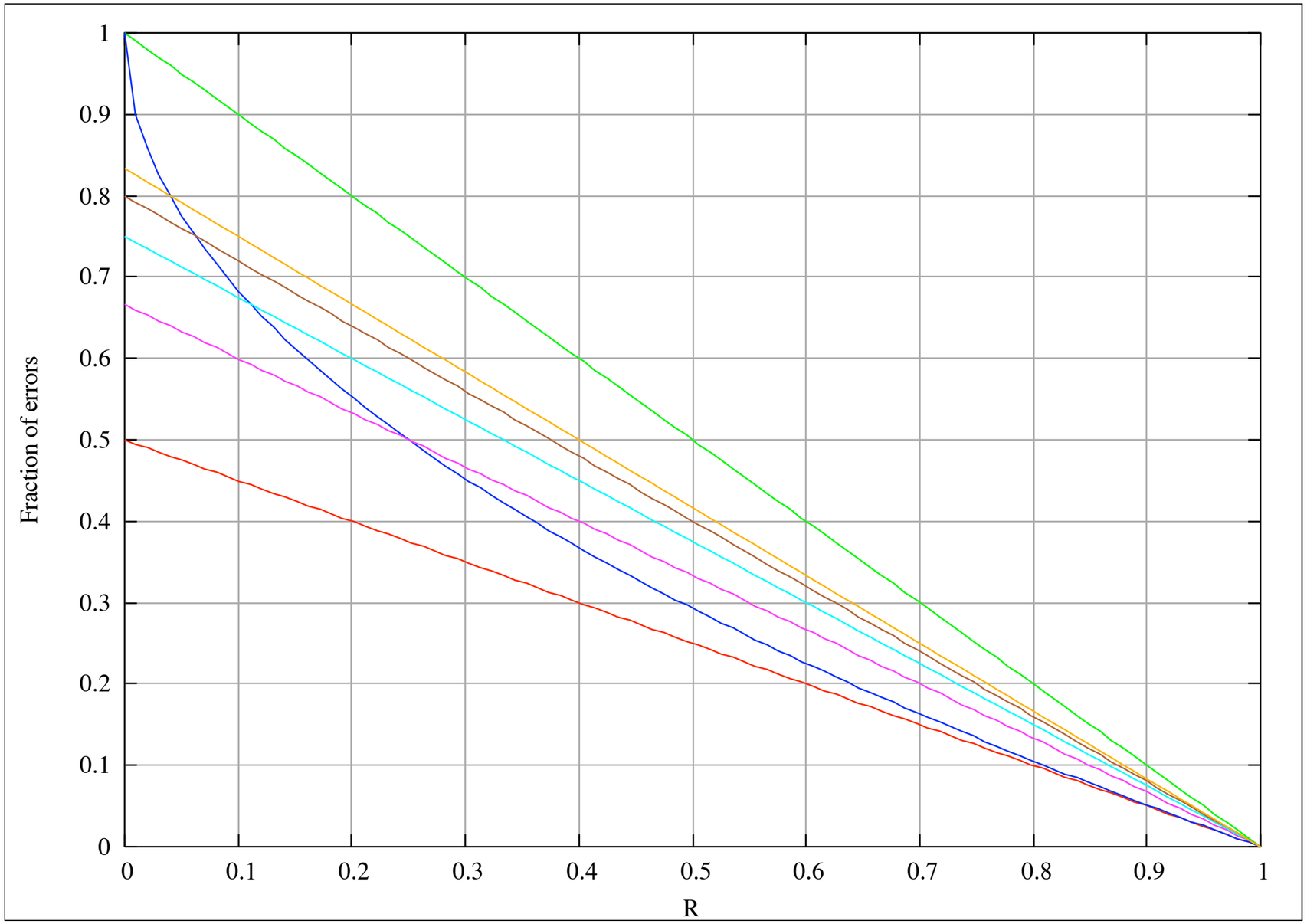

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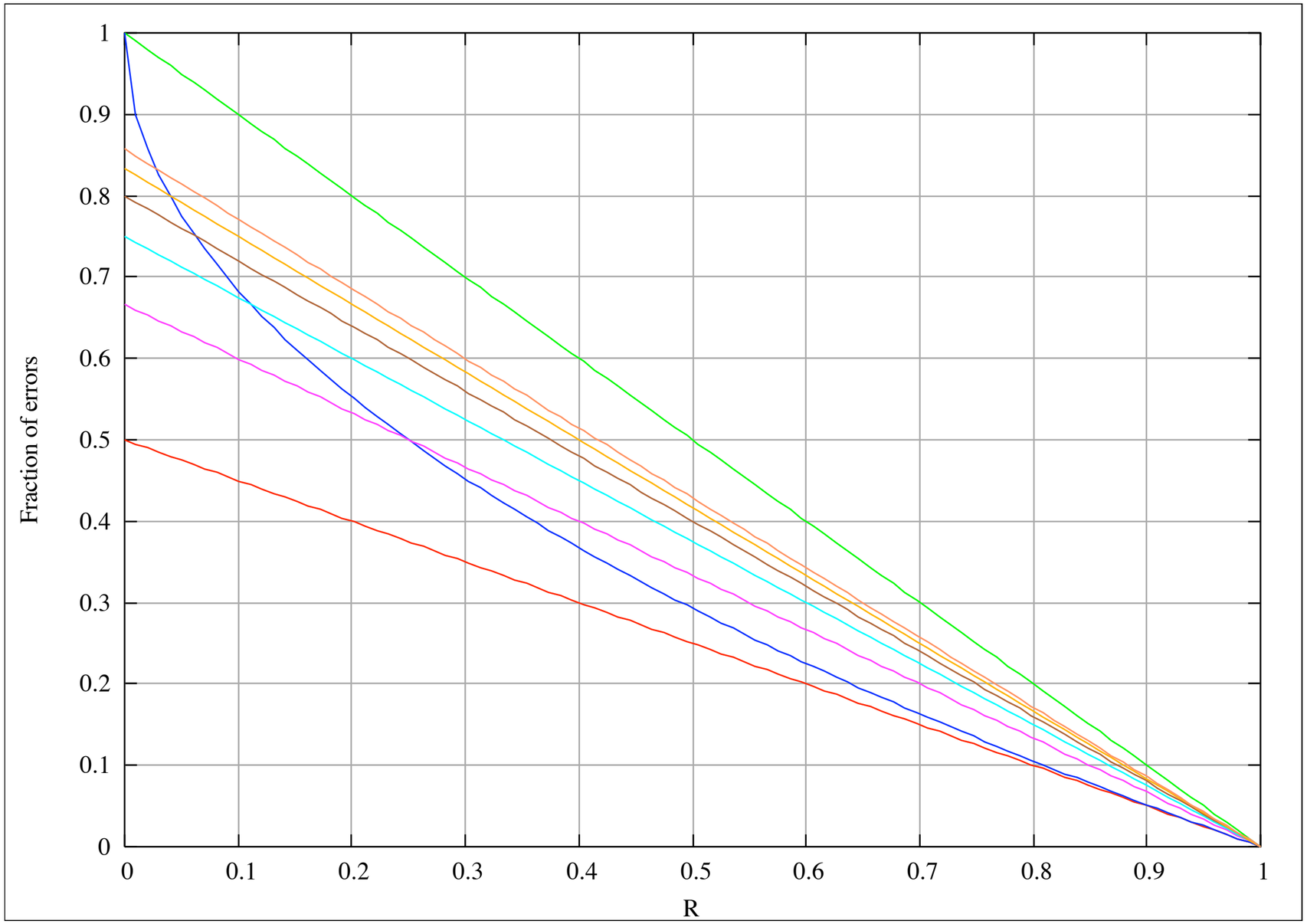

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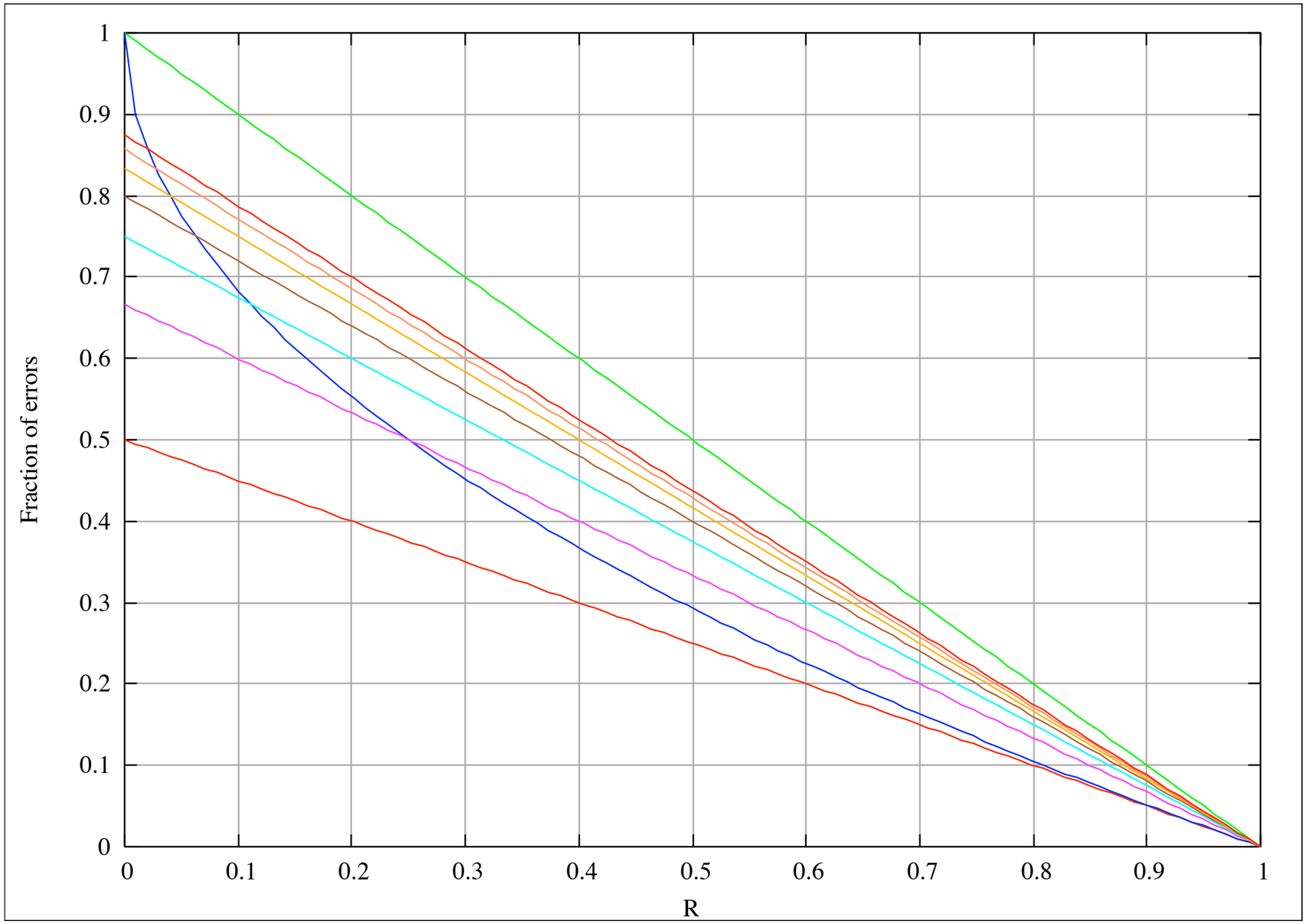

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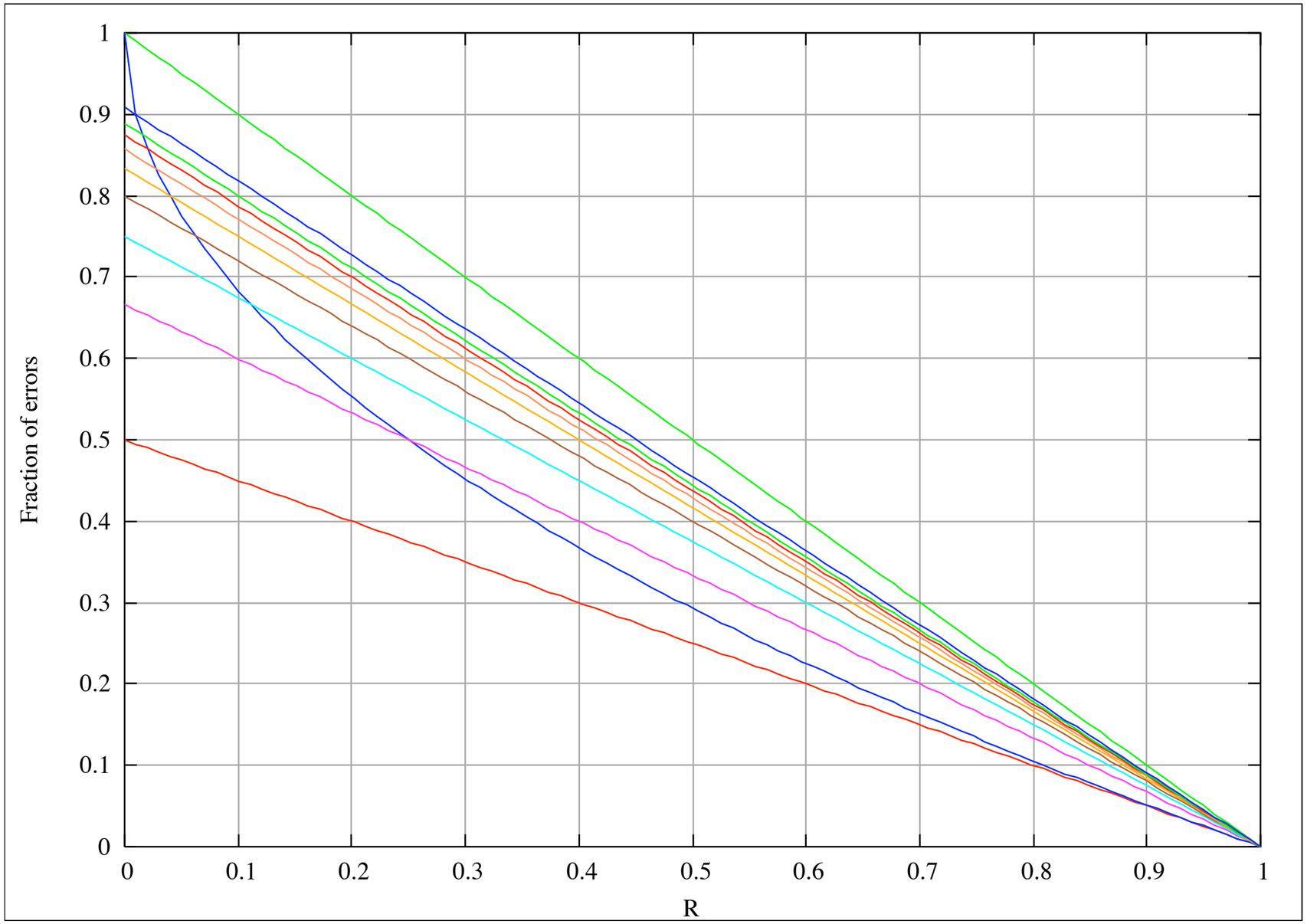

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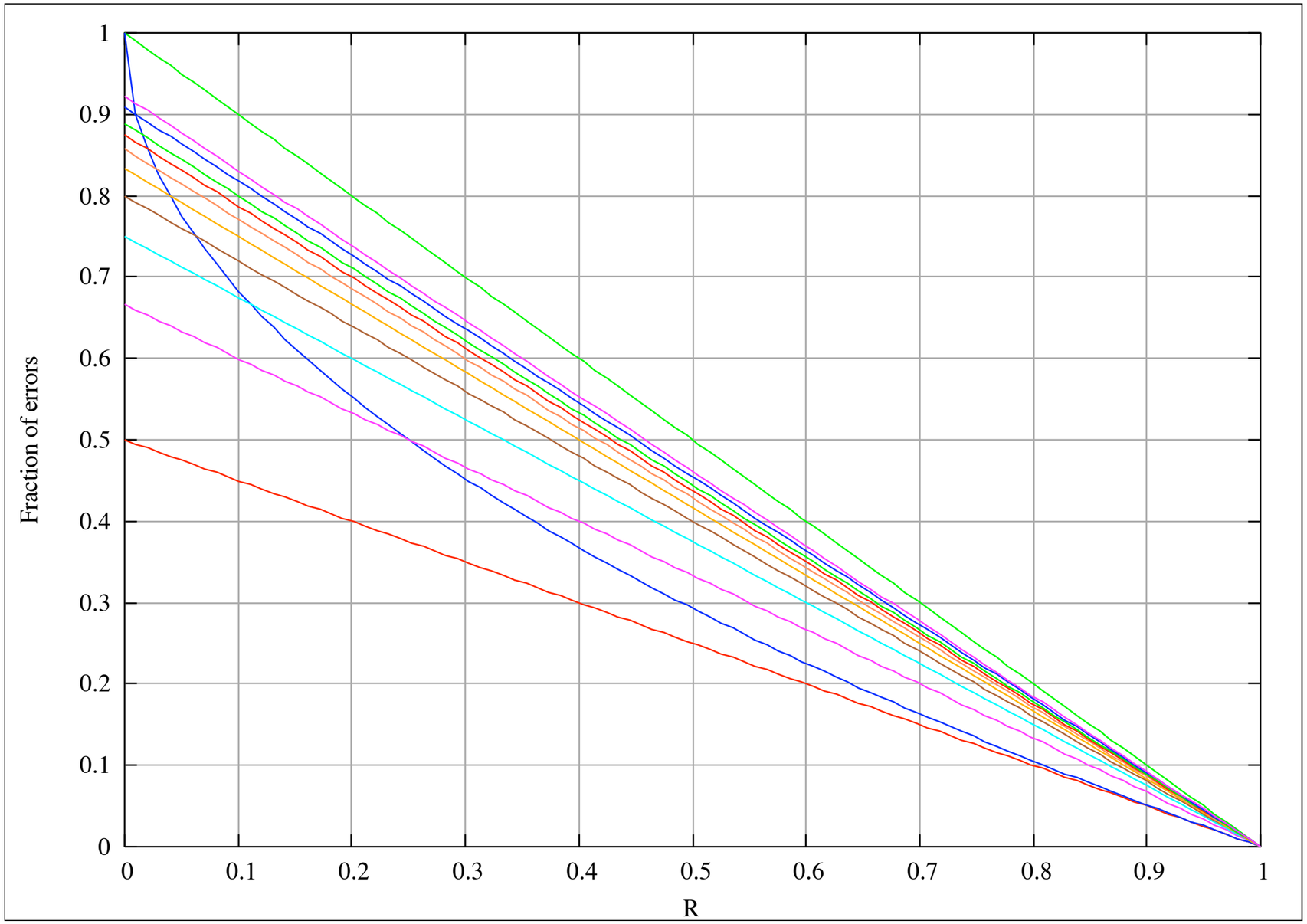

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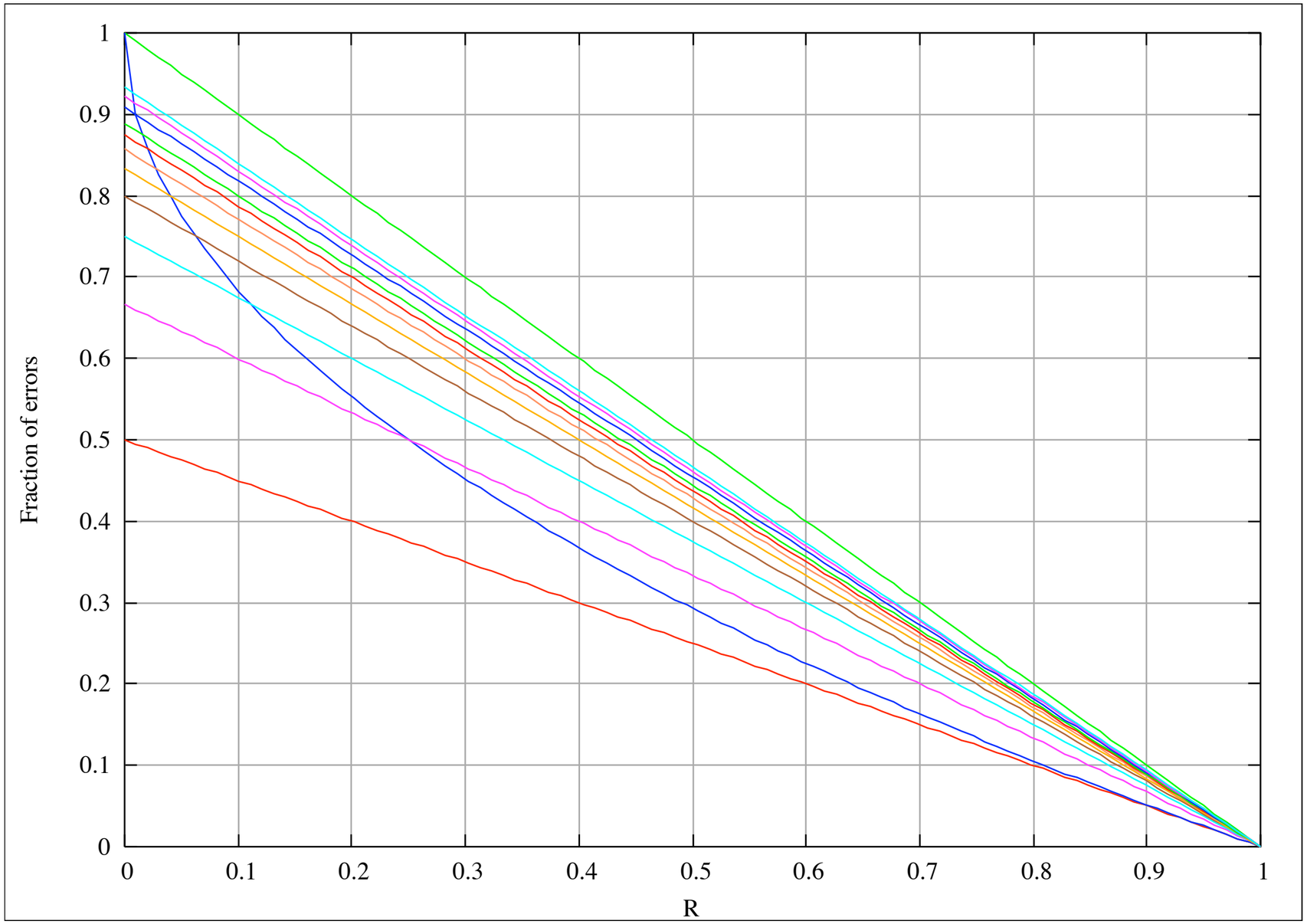

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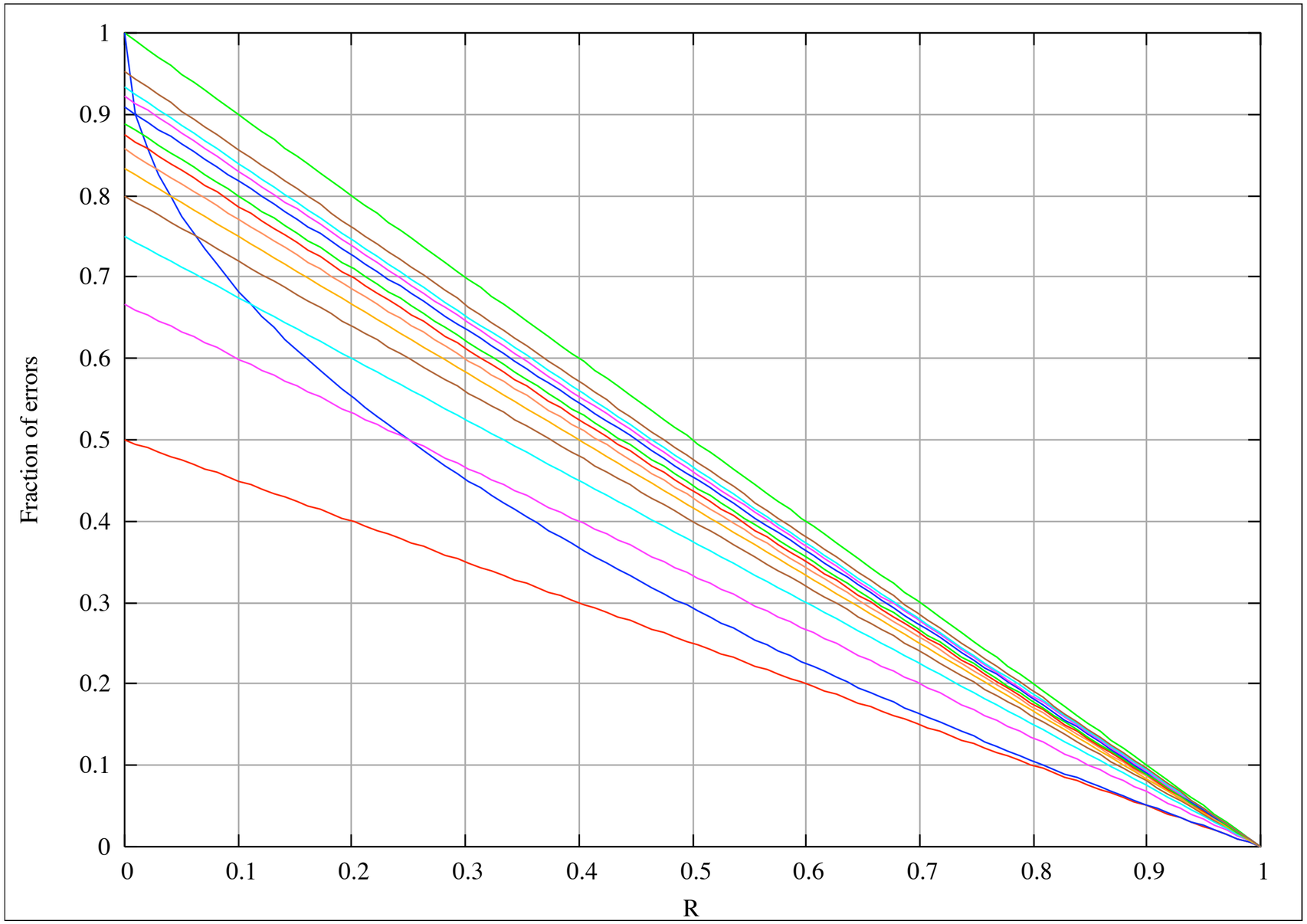

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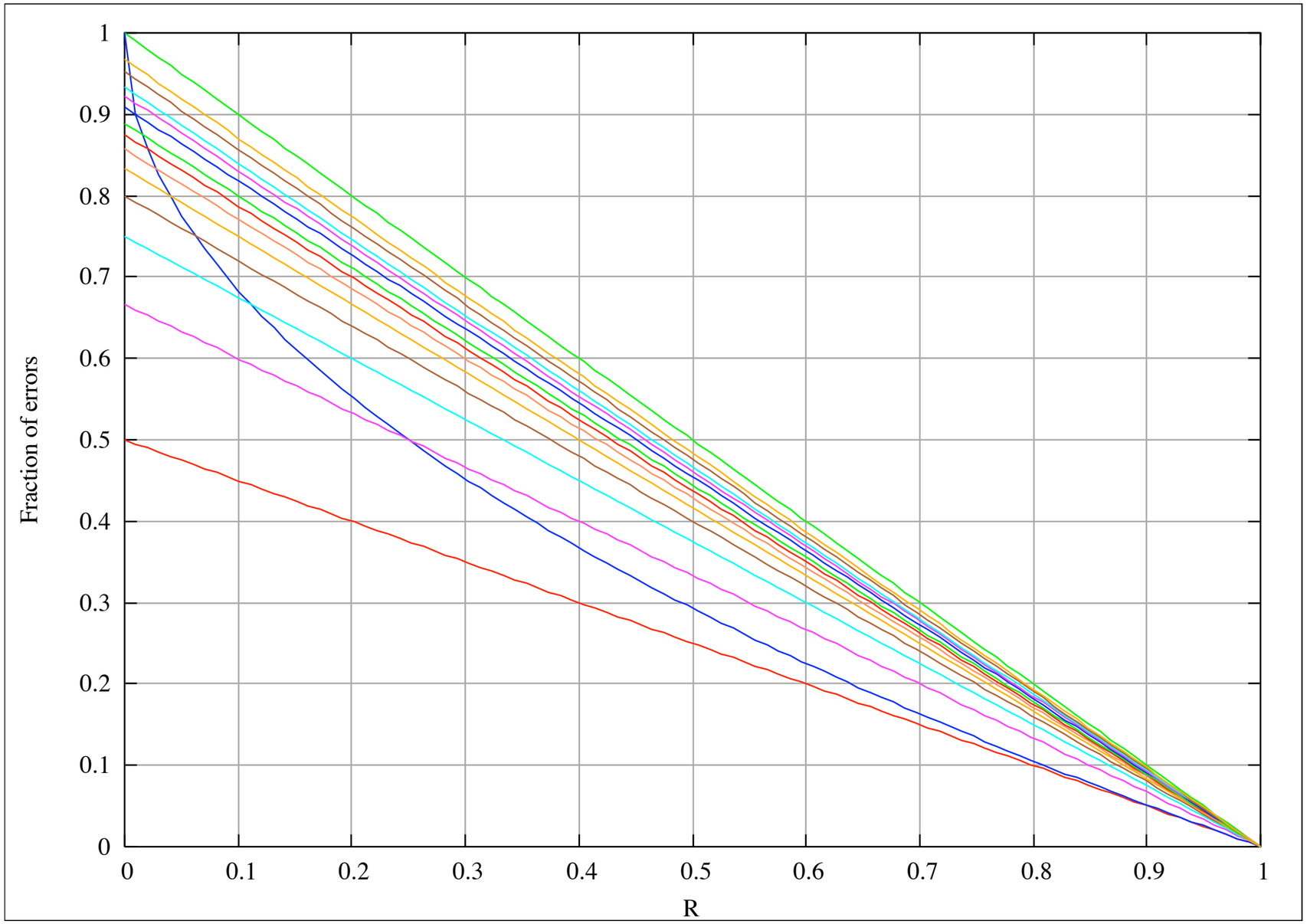

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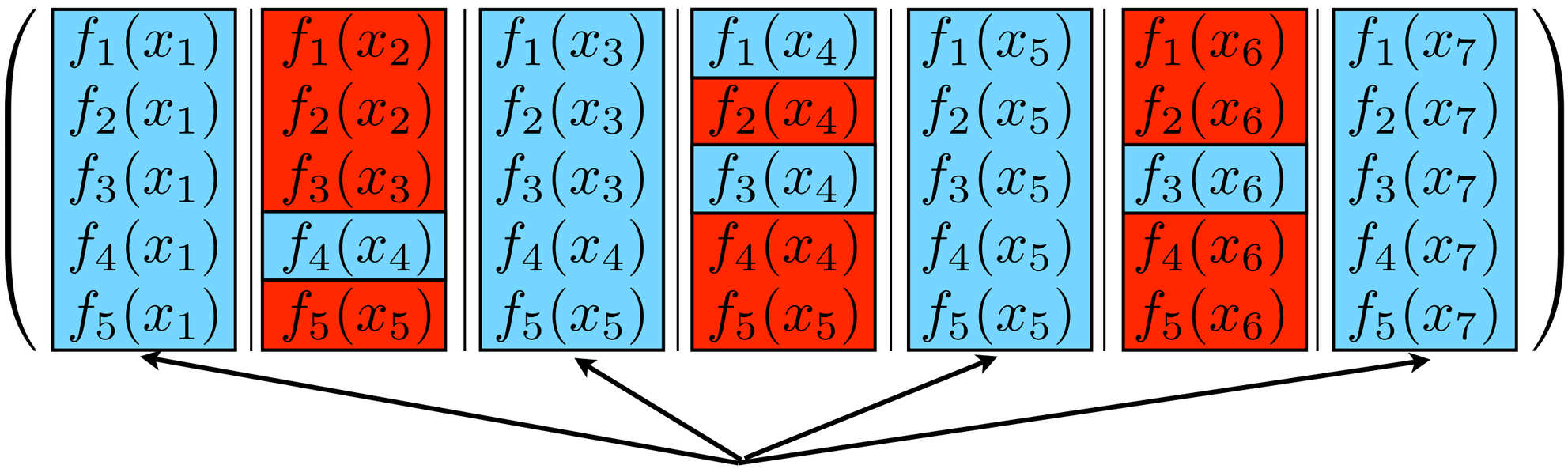
$$\left(\begin{array}{c|c|c|c} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_m(x_1) & f_m(x_2) & \cdots & f_m(x_n) \end{array}\right)$$

Interleaved codeword

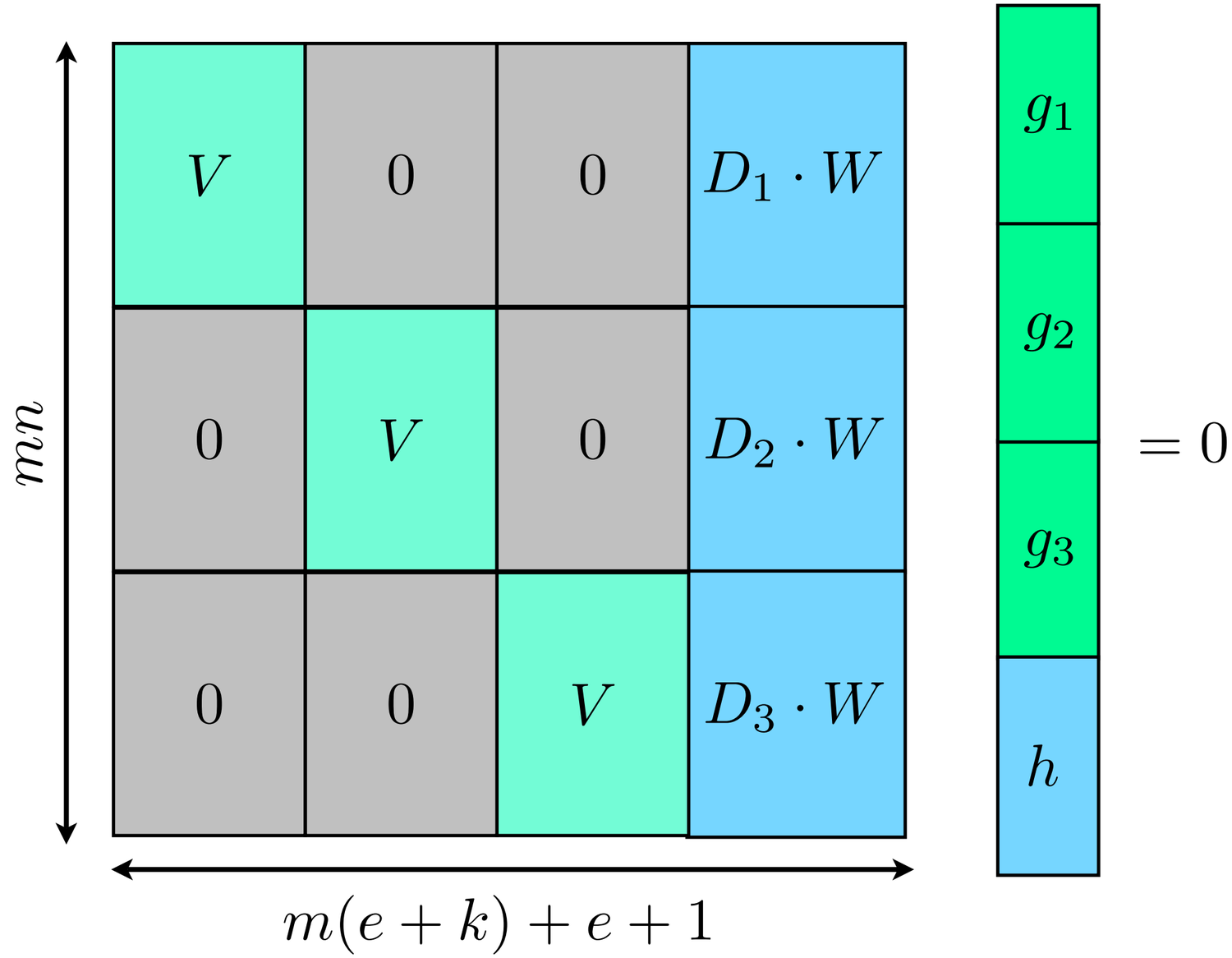
RS-Codes over Large Alphabets



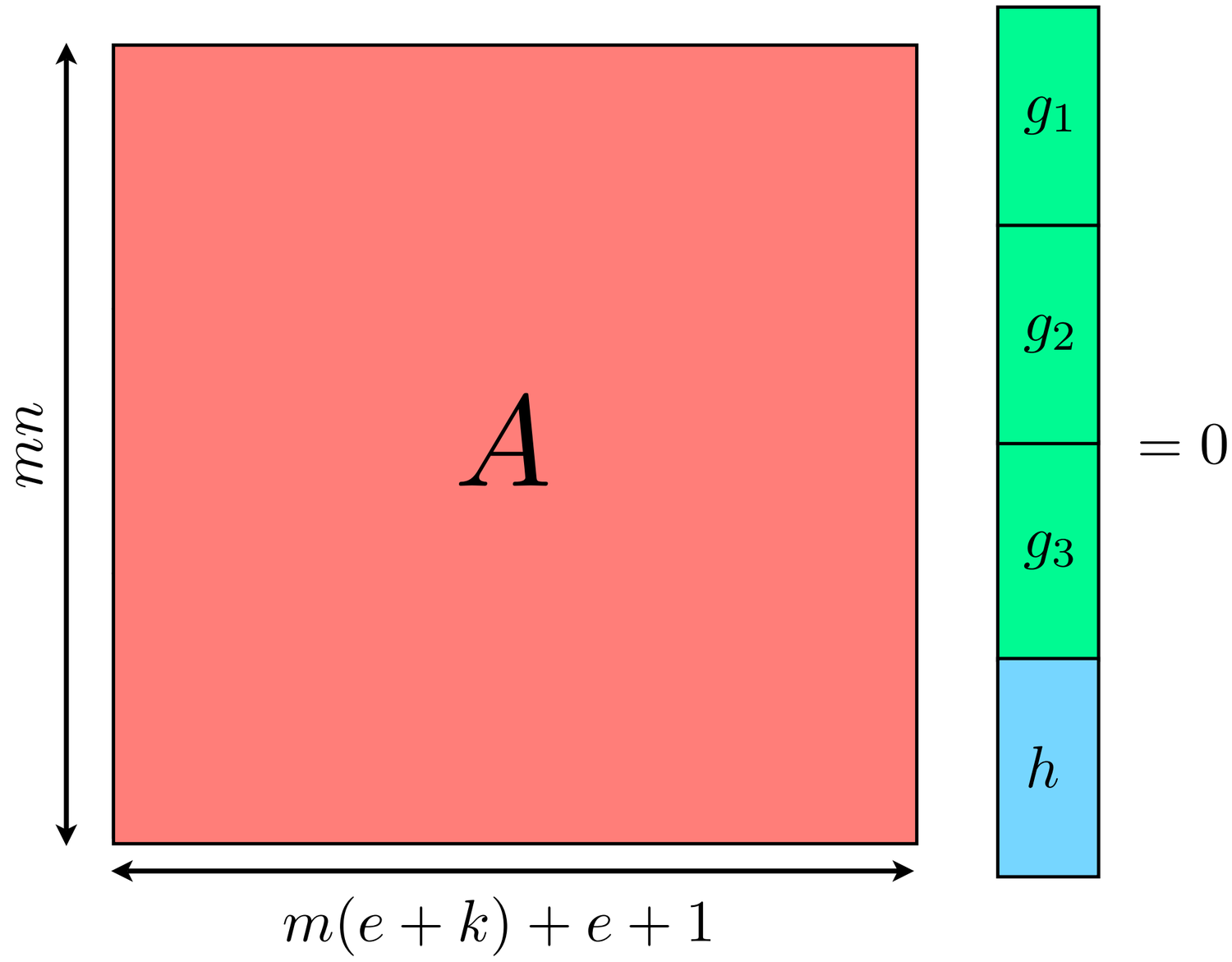
Non-error positions are the same for all the polynomials, so same error locator

$$
\begin{aligned}
g_1(x_i) - y_{i,1}h(x_i) &= 0 \\
g_2(x_i) - y_{i,2}h(x_i) &= 0 \\
g_3(x_i) - y_{i,3}h(x_i) &= 0 \\
g_4(x_i) - y_{i,4}h(x_i) &= 0 \\
g_5(x_i) - y_{i,5}h(x_i) &= 0
\end{aligned}
$$

RS-Codes over Large Alphabets



RS-Codes over Large Alphabets



Algorithm

$$A \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ h \end{pmatrix} = 0$$

1. Find element $(g_1|g_2|g_3|h)^\top$ in the right kernel of A.
2. If $g_i/h \notin \mathbb{F}_q[x]_{<k}$ for some i, return error.
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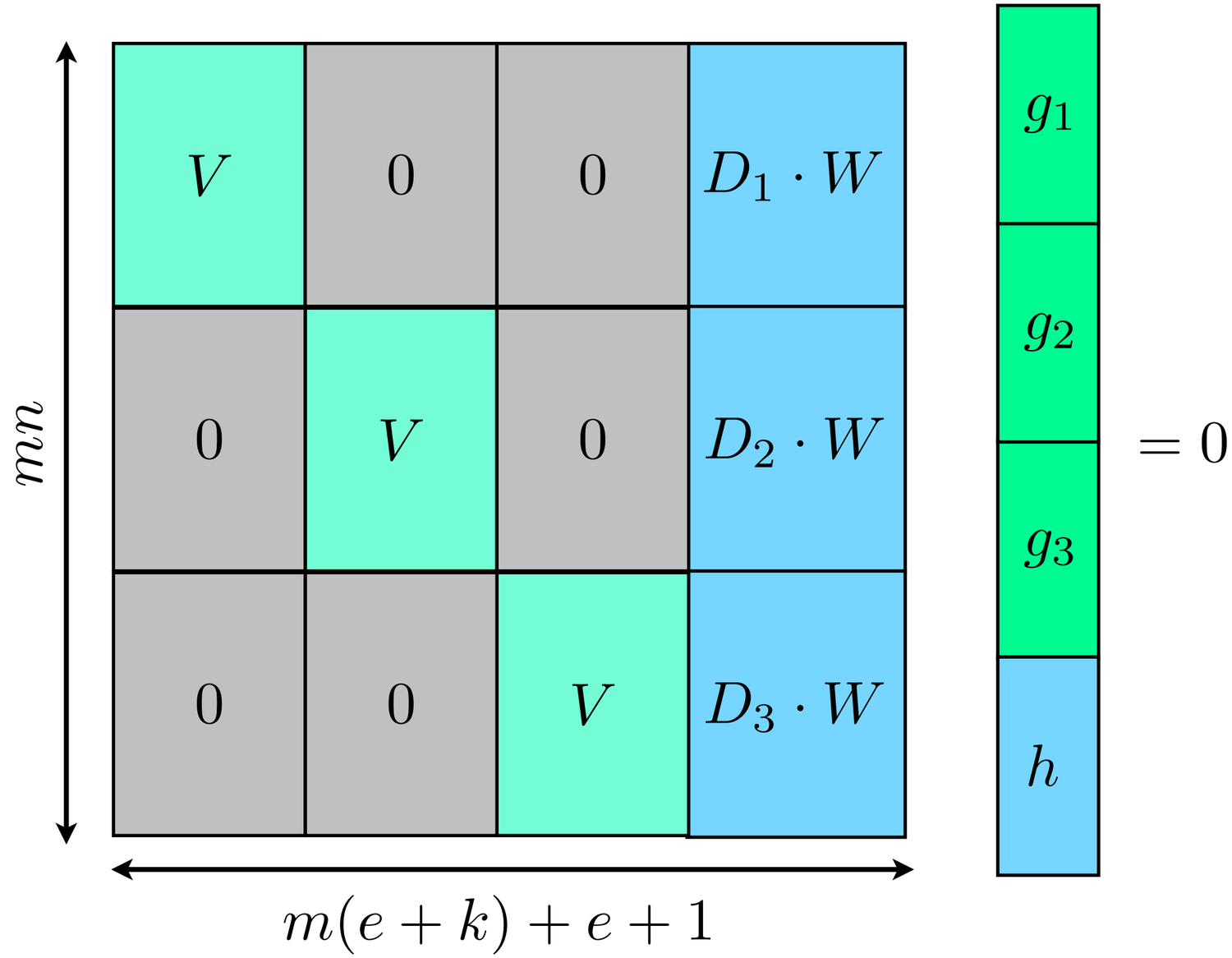
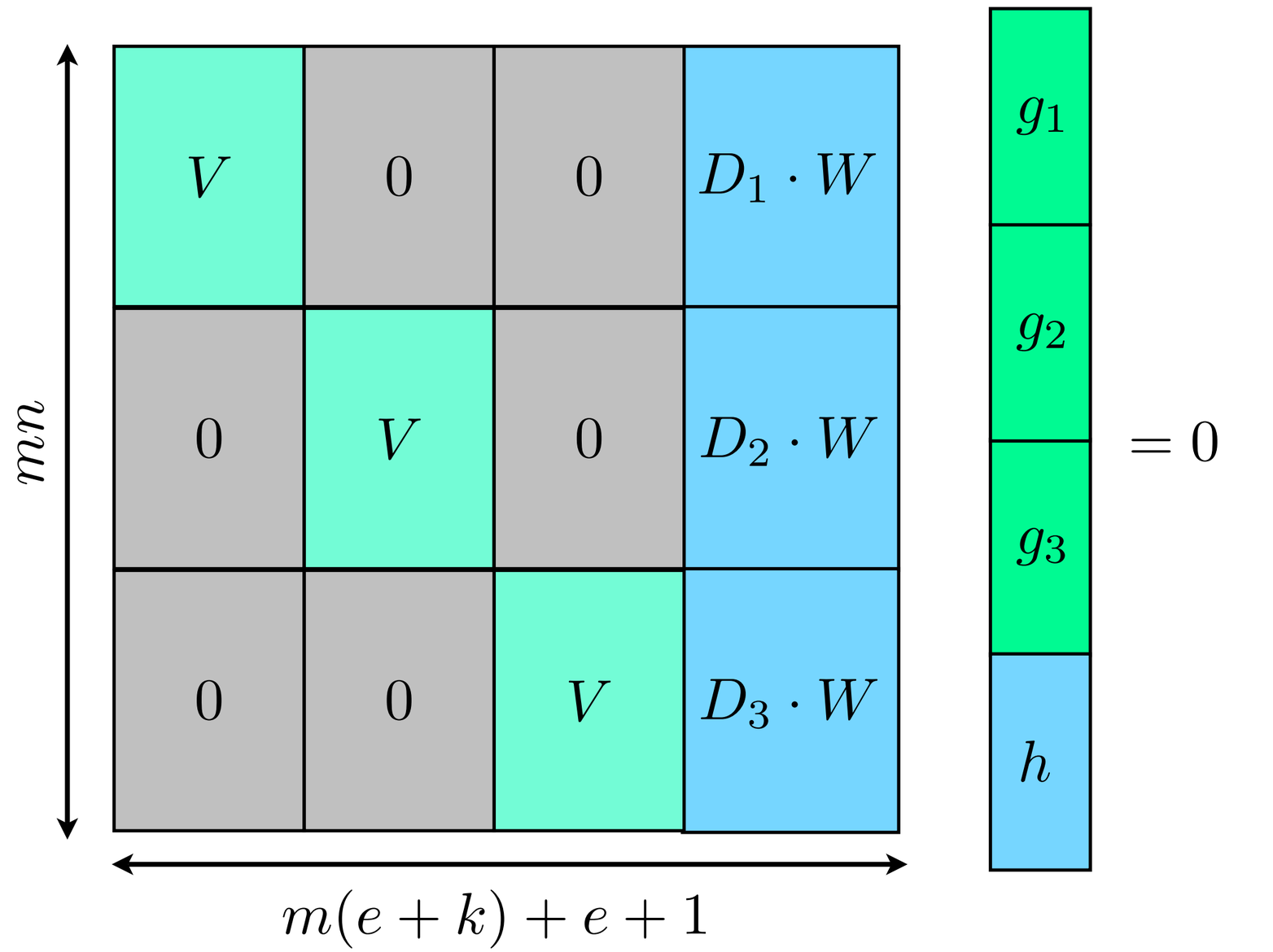
RS-Codes over Large Alphabets



Right kernel is one-dimensional $\Rightarrow$

$$m(e+k)+e+1 = n+1 \Rightarrow \boxed{e = \frac{m}{m+1}(n-k)}$$

Problem



Uniqueness is not guaranteed!

BKY Model

1. Choose $e \leq \frac{m}{m+1}(n-k)$ positions.
2. For each of these positions at least one of the components is in error.
3. The error value is uniformly distributed in $\mathbb{F}_q$.

$$\left(\begin{array}{c|c|c|c|c|c|c} f_1(x_1) & f_1(x_2) & f_1(x_3) & f_1(x_4) & f_1(x_5) & f_1(x_6) & f_1(x_7) \\ f_2(x_1) & f_2(x_2) & f_2(x_3) & f_2(x_4) & f_2(x_5) & f_2(x_6) & f_2(x_7) \\ f_3(x_1) & f_3(x_3) & f_3(x_3) & f_3(x_4) & f_3(x_5) & f_3(x_6) & f_3(x_7) \\ f_4(x_1) & f_4(x_4) & f_4(x_4) & f_4(x_4) & f_4(x_5) & f_4(x_6) & f_4(x_7) \\ f_5(x_1) & f_5(x_5) & f_5(x_5) & f_5(x_5) & f_5(x_5) & f_5(x_6) & f_5(x_7) \end{array}\right)$$

What is the probability of decoding error?

BKY Model

Bleichenbacher et al.: If $e = \frac{m}{m+1}(n-k)$, then the probability of decoding error is $O(n/q)$.

Brown et al.: If $e = \frac{m}{m+1}(n-k)$, then the probability of decoding error is $O(1/q)$.

This talk: Roughly, if $e = \frac{m}{m+1}(n-k) - \epsilon n$, then the error probability of the decoder is $O(q^{-m\epsilon n})$.

Comparison to Other Methods

Hashing: every symbol has $q^{1-\delta}$ symbols; error probability is $O(n/q^{\delta})$.

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Coppersmith and Sudan: Different algorithm, error probability is $O(q^{O(m)}/n)$.

Hashing and error-correction: slightly worse rate with same error probability as in our case.

BKY Model

Suppose that

$$e \leq \frac{\beta m}{\beta m + 1}(n-k) - \frac{c}{\beta m + 1},$$

for some $c > 0$, and where $\beta = \frac{\ln(q^m - 1)}{\ln(q^m)}$. Then we have:

(1) If $e + t < n - k$, then the error probability of the decoder is zero.

(2) In general the error probability of the decoder is at most $\frac{q}{q-1} \cdot q^{-c}$.

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$$\beta \simeq 1$$

$$\text{Roughly: } e \leq \frac{m}{m+1}(n-k) - \epsilon n \Rightarrow \text{Error probability } \simeq q^{-m\epsilon n}$$

Method of Proof

$$A \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ h \end{pmatrix} = 0$$

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3. If not, then set $f_i = g_i/h$.

Assumptions

(a) The error positions are $1, 2, \ldots, e$.

(b) The functions $f_1, \ldots, f_m$ sent over the channel are all zero.

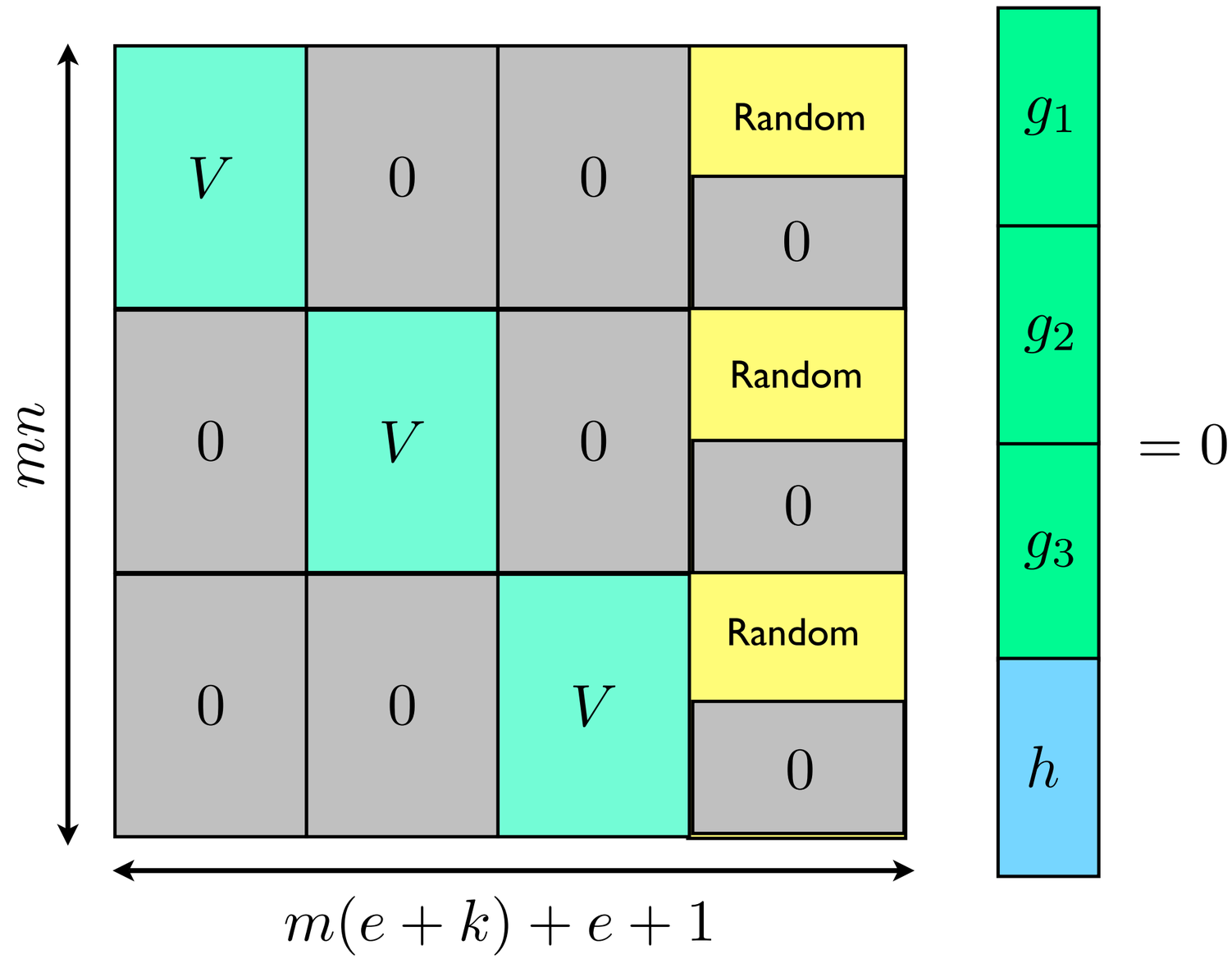
Assumption (a) amounts to re-ordering and is trivial.

Assumption (b) follows from linearity:
Error probability is independent of the choice of the word sent.

Assumptions

$$
\underbrace{\begin{pmatrix}
V & 0 & 0 & D_1 \cdot W \\
0 & V & 0 & D_2 \cdot W \\
0 & 0 & V & D_3 \cdot W
\end{pmatrix}}_{m(e+k)+e+1}
\underbrace{\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ h \end{pmatrix}}_{uu}
= 0
$$

Assumptions



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There exists $(g_1|g_2|g_3|h)$ in the right kernel of A such that $g_j \neq 0$ for some j.

Method of Proof

$$A \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ h \end{pmatrix} = 0$$

Given $v = (g_1|g_2|g_3|h)$, with some $g_j \neq 0$, calculate $\Pr[v \in \ker(A)]$

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Use Markov's inequality

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Hard part

Complexity

$$A \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ h \end{pmatrix} = 0$$

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$$O((mn) \log^2(mn) \log^2(q))?$$

An Example

Want to transmit an MP3 file (4MB) over a packet-based network to a receiver, and every packet is 1024 bytes.

Due to low level of physical layer error-correction, we expect that at most 33% of the packets are corrupted.

Want to be able to recover the file with high probability using packet level error correction.

An Example

$$q = 2^{16} \qquad \Rightarrow \quad \beta \simeq 1$$

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$$m = 512$$

$$k = 4000 \qquad e \leq \frac{n}{3} = \frac{512}{513}(n-k) - 2.75$$
$$n = 6010$$

$$\text{Error probability} \leq q^{-2.75m} < 10^{-6000}$$

AG-Codes

For a variety of reasons, we are interested in constructing codes for which q and m are fixed, but the length goes to infinity.

Not possible for RS-codes: the length of the code is no more than q+1.

AG-codes offer a solution! They are constructed by

(a) Realizing the RS-codes are related to the projective line, and
(b) replacing the projective line with an algebraic curve.

The algorithm described has to be properly generalized to AG-codes, and the new algorithm has to be analyzed in this setting.

Refer to the paper to see how this is done!

AG-Codes

In this case the upper bound on the number of correctable errors is

$$\frac{\beta m}{\beta m + 1}(n - k) - 2g$$

where g is the genus of the curve. The probabilistic statements remain with respect to this bound.

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Cirencester, December 2005

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2. Provided a new analysis which guarantees exponentially small error probabilities.

3. Provided generalizations to AG-codes.

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Open Questions

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2. Is it possible to improve the running time of the algorithms to

$$O(mn^{1+\epsilon}\log^{1+\epsilon}(q))?$$

3. Is it possible to make the algorithms "practical?"

4. Are there other applications of "Interleaved codes?"

~~Open~~ Questions

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Thank you.

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