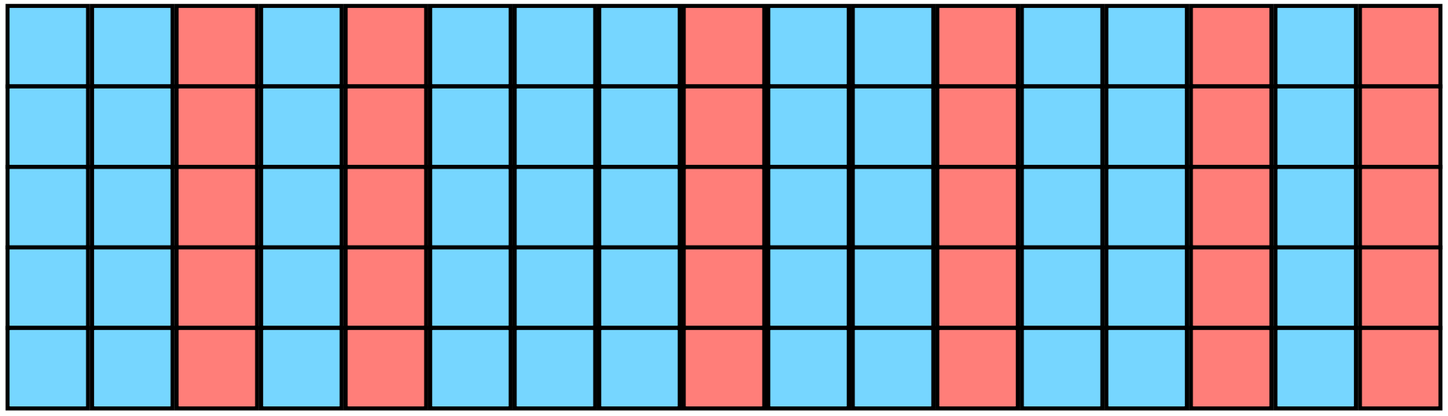
(Improved) Decoding of Interleaved RS- and AG-Codes



Amin Shokrollahi

Joint work with A. Brown and L. Minder

EPFL

Diamantina, August 2006

RS-Codes

$x_1, \ldots, x_n \in \mathbb{F}_q$ pairwise distinct

$\mathbb{F}_q[x]_{<k}$ space of polynomials of degree $<k, k \leq n$

$$\begin{array}{rcl} \varphi\colon \mathbb{F}_q[x]_{<k} & \longrightarrow & \mathbb{F}_q^n \\ f & \longmapsto & (f(x_1), \ldots, f(x_n)) \end{array}$$

$\mathrm{Im}(\varphi)$ is called a RS-code $\mathcal{C}$

A nonzero polynomial of degree $<k$ over a field has at most k-1 roots over the field.

$$\ker(\varphi) = \{f \mid f(x_1) = \cdots = f(x_n) = 0\} = 0.$$

$$0 \neq f \in \mathbb{F}_q[x]_{<k} \Rightarrow \#\{i \mid f(x_i) = 0\} < k \Rightarrow \text{Minimum distance of } \mathcal{C} \geq n-k+1.$$

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$$\mathcal{C} \text{ is } [n, k, n-k+1]_q - \text{code}$$

Decoding Problem

$$(\quad f(x_1) \quad , \ldots , \quad f(x_n) \quad)$$

Diamantina, August 2006

Decoding Problem

$$\begin{pmatrix} f(x_1) & , \ldots , & f(x_n) \end{pmatrix}$$

Transmission

$$\begin{pmatrix} y_1 & , \ldots , & y_n \end{pmatrix}$$

Diamantina, August 2006

Decoding Problem

$$\underset{\text{Known}}{(\quad y_1 \quad ,\ldots,\quad y_n \quad)} \xleftarrow{\text{Transmission}} \underset{\text{Unknown}}{(\quad f(x_1) \quad ,\ldots,\quad f(x_n) \quad)}$$

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Error positions $E := \{i \mid y_i \neq f(x_i)\}$

Unique Decoding Problem

$$\underset{\text{Known}}{(\quad y_1 \quad ,\ldots,\quad y_n \quad)} \xleftarrow{\text{Transmission}} \underset{\text{Unknown}}{(\quad f(x_1) \quad ,\ldots,\quad f(x_n) \quad)}$$

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Promise: $\#E =: e \leq t$ for some t.

Decoding Problem

$$\underbrace{(\quad f(x_1) \quad ,\ldots, \quad f(x_n) \quad)}_{\text{Unknown}}$$

Transmission

$$\underbrace{(\quad y_1 \quad ,\ldots, \quad y_n \quad)}_{\text{Known}}$$

Error positions $E := \{i \mid y_i \neq f(x_i)\}$

Promise: $\#E =: e \leq t$ for some t.

Decoding problem: find f.

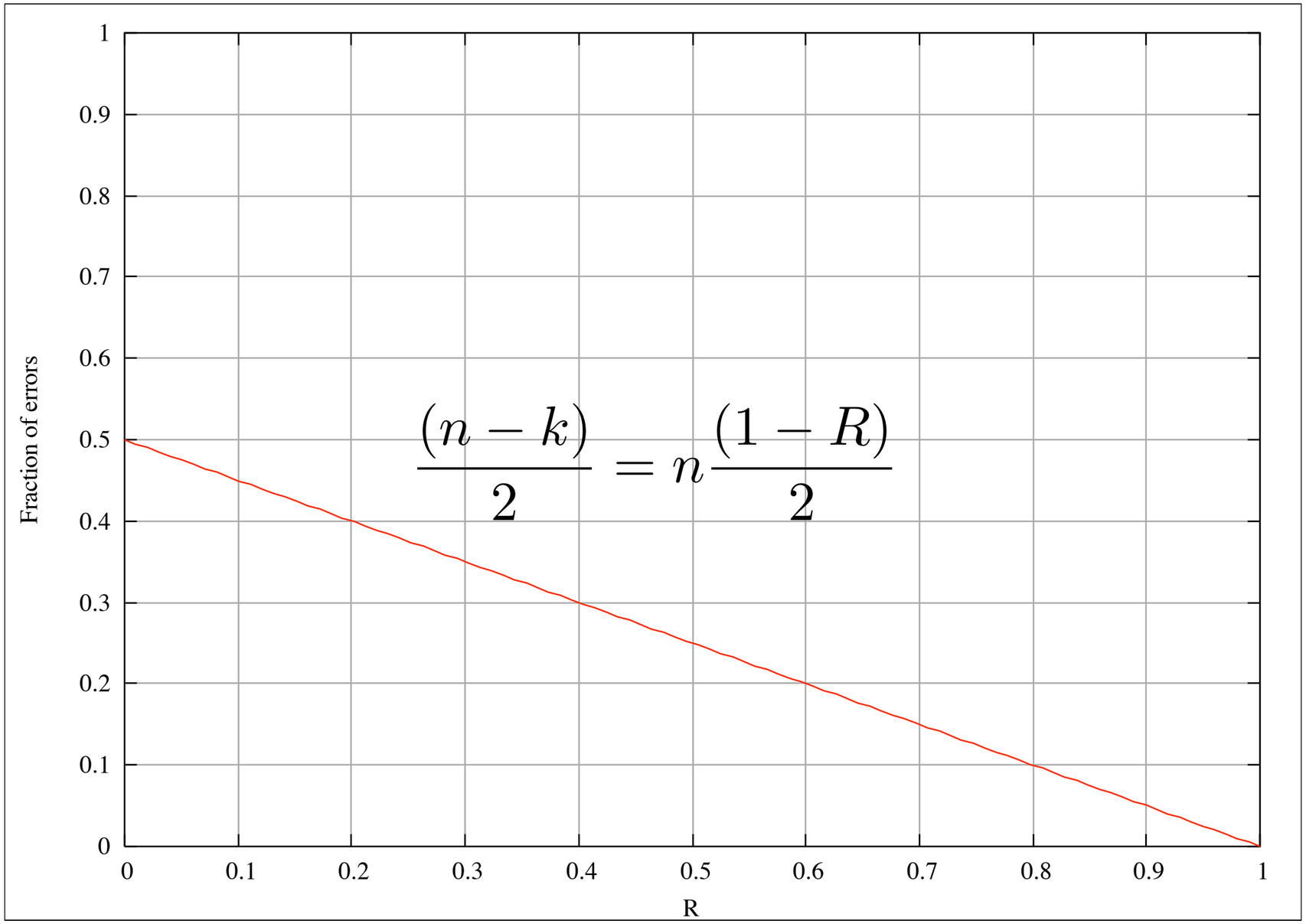
How Many Errors Can We Correct?

Berlekamp, Welch-Berlekamp: $\frac{t}{n} = \frac{1-R}{2}$

Sudan, Guruswami-Sudan (list-decoding): $\frac{t}{n} = 1 - \sqrt{R}$

What is the ultimate bound? $\frac{t}{n} = 1 - R$

Decoding Errors



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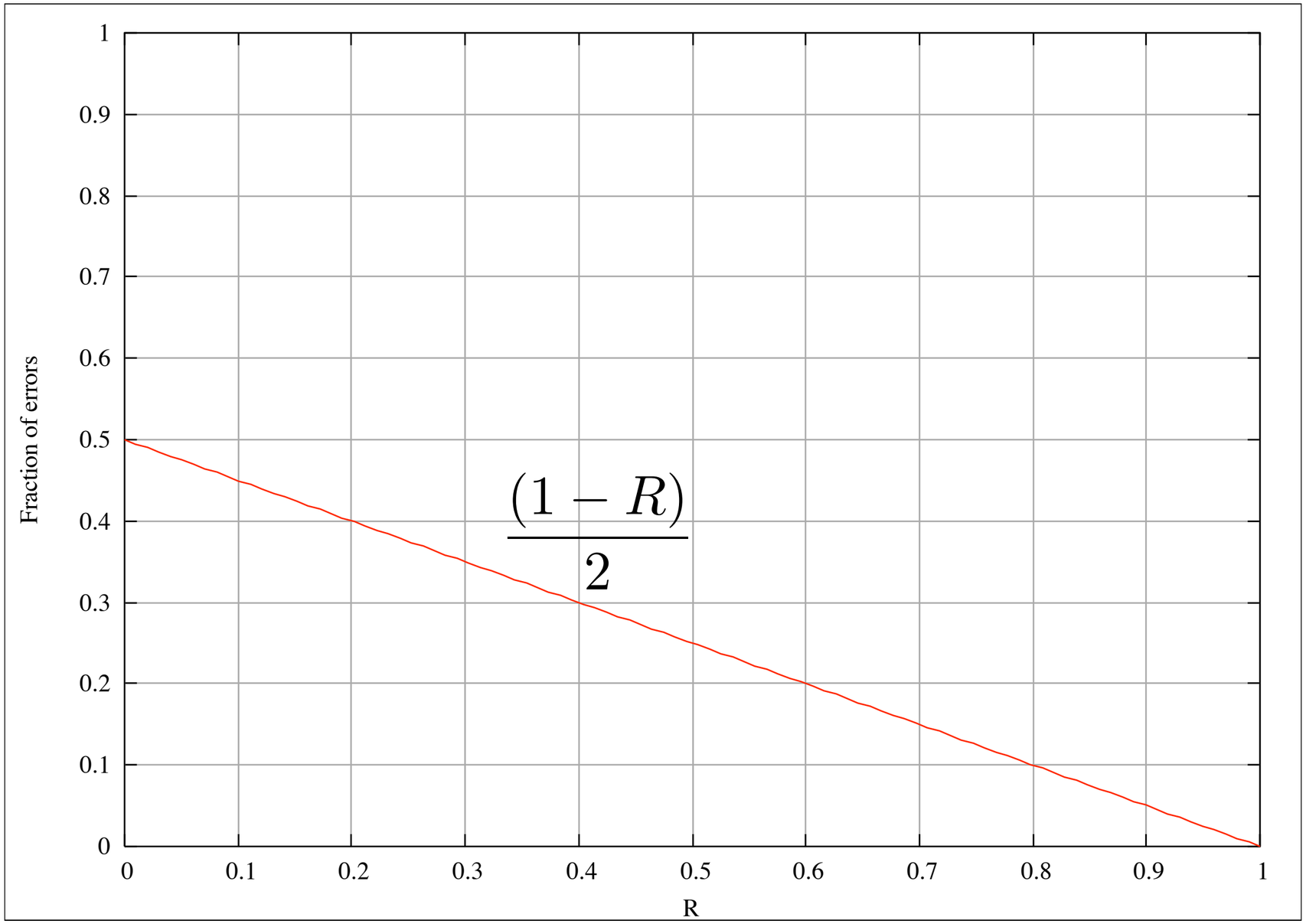

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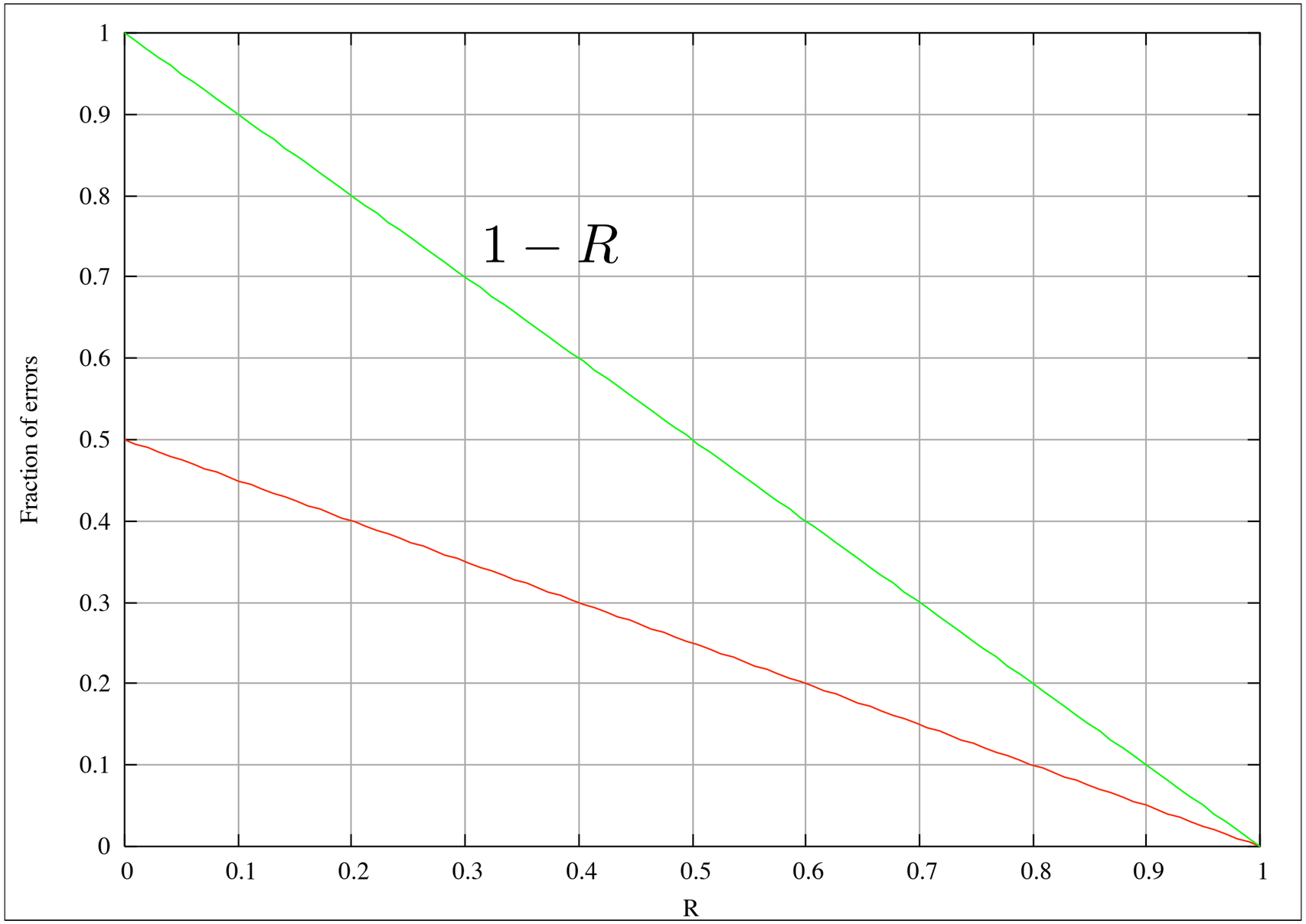

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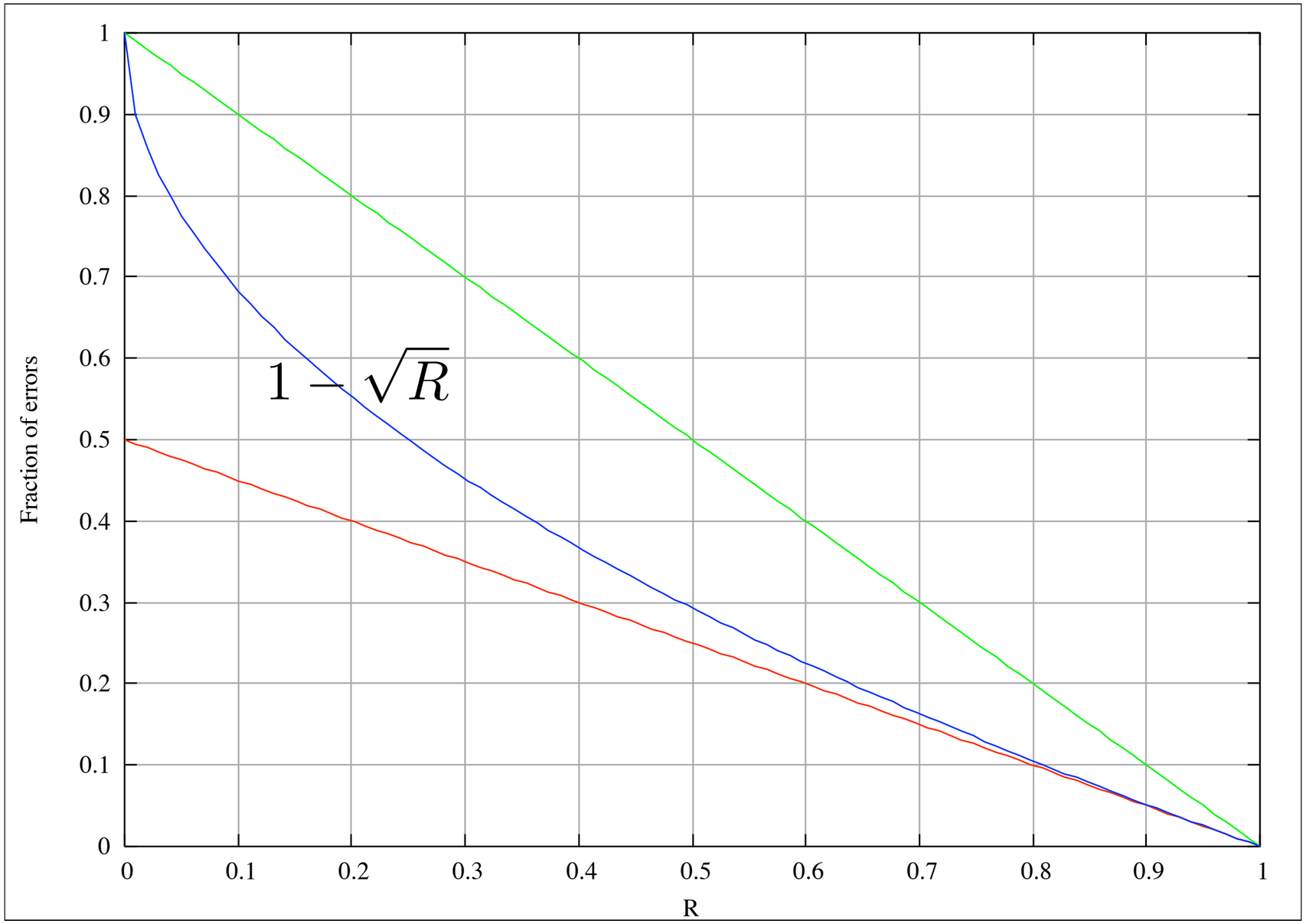


Decoding More Errors?

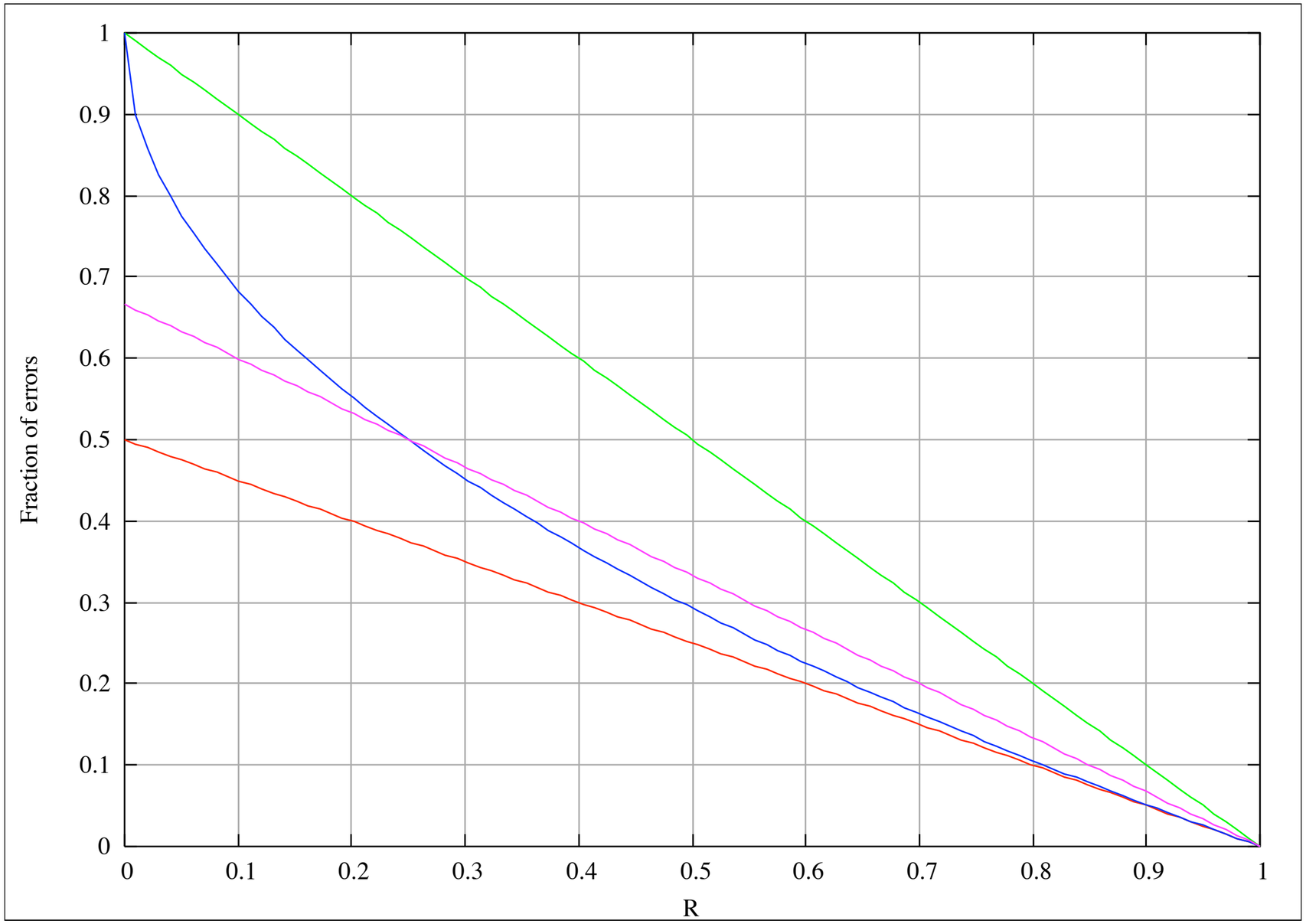
We will introduce probabilistic algorithms parameterized by a parameter m.

Algorithms will be able to correct (roughly) up to

$$\frac{t}{n} = \frac{m}{m+1}(1-R)$$

The probabilistic model and main parts of the algorithm are due to Bleichenbacher, Kiyaias, and Yung (BKY).

Decoding More Errors?



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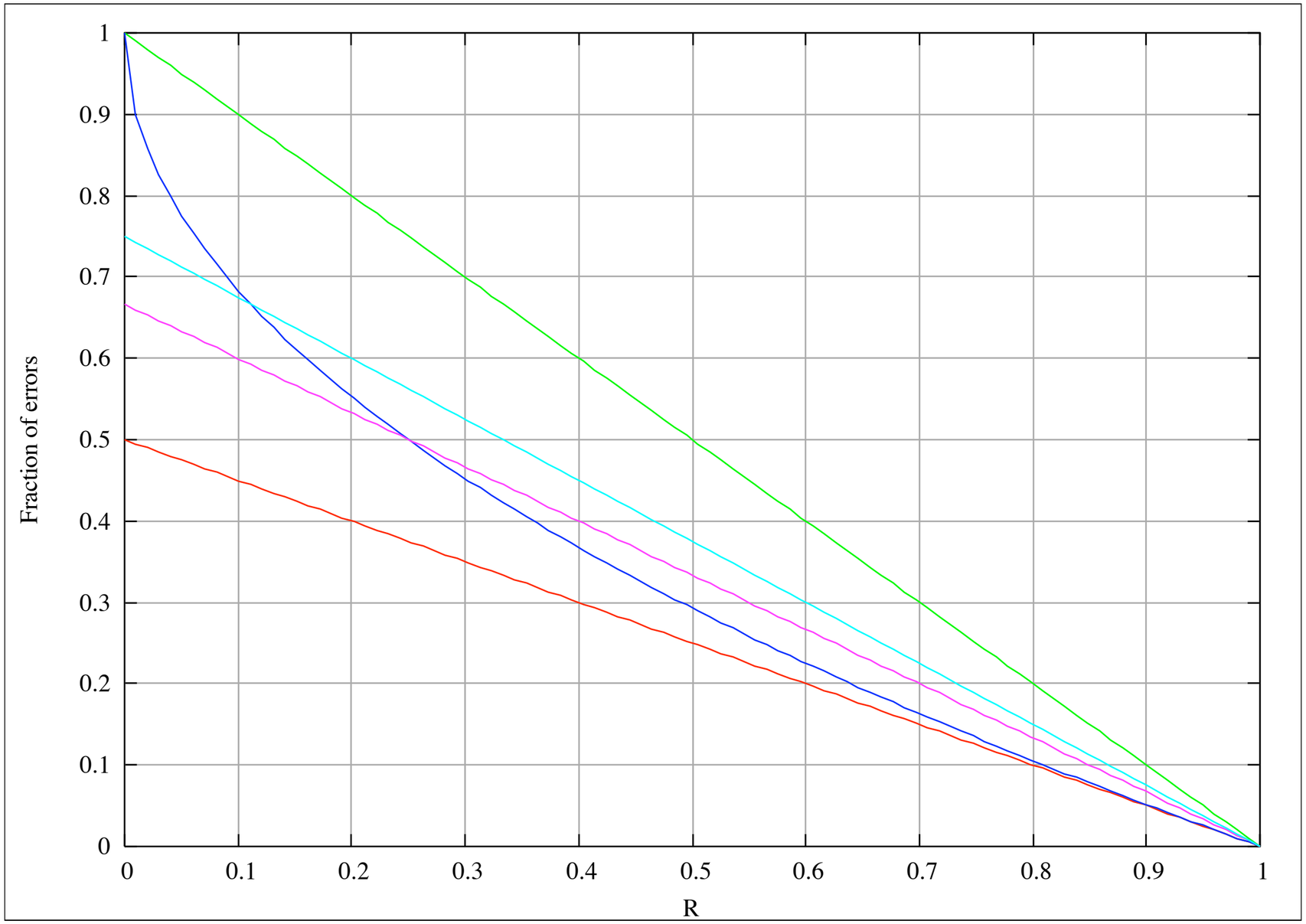

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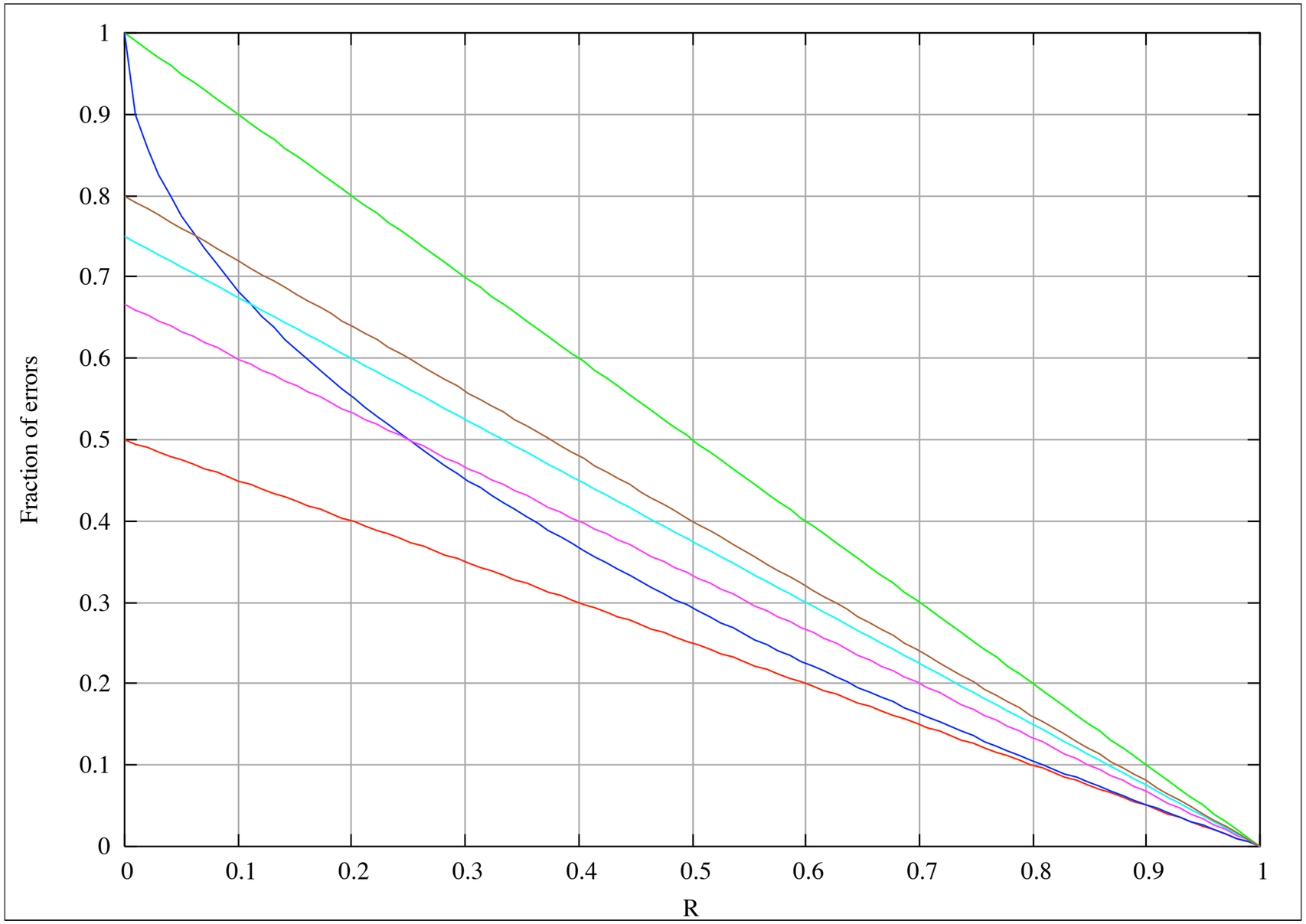

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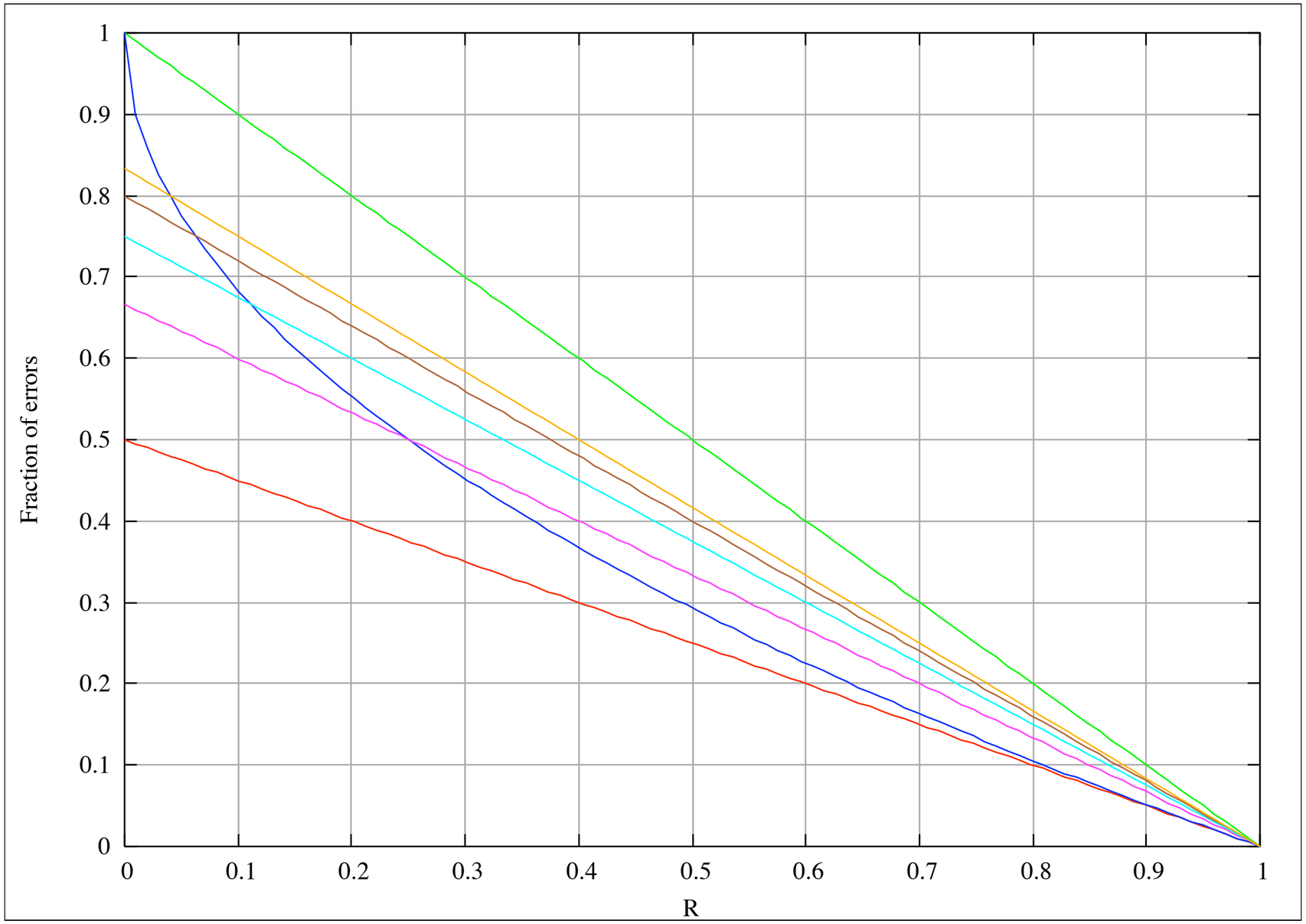

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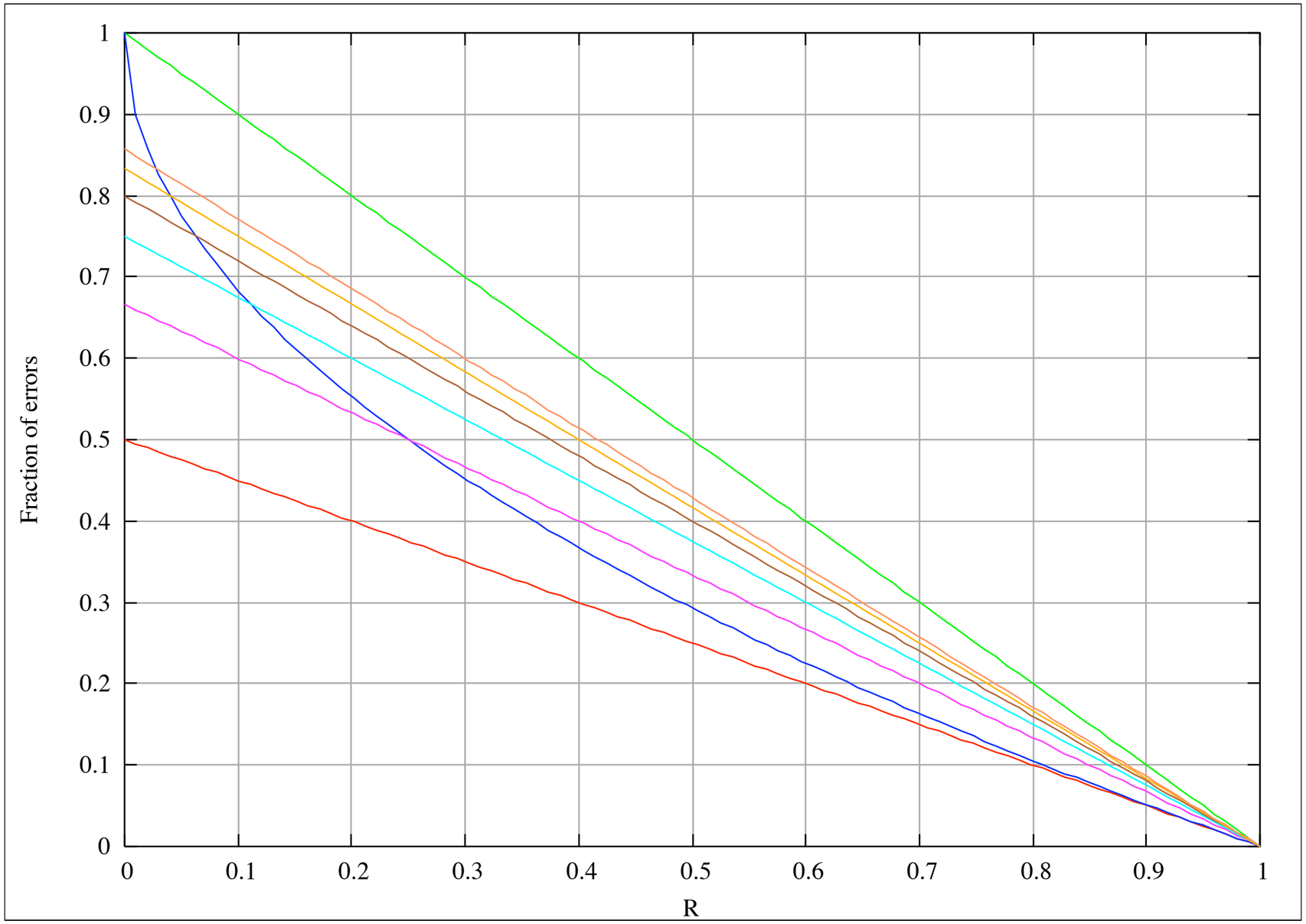

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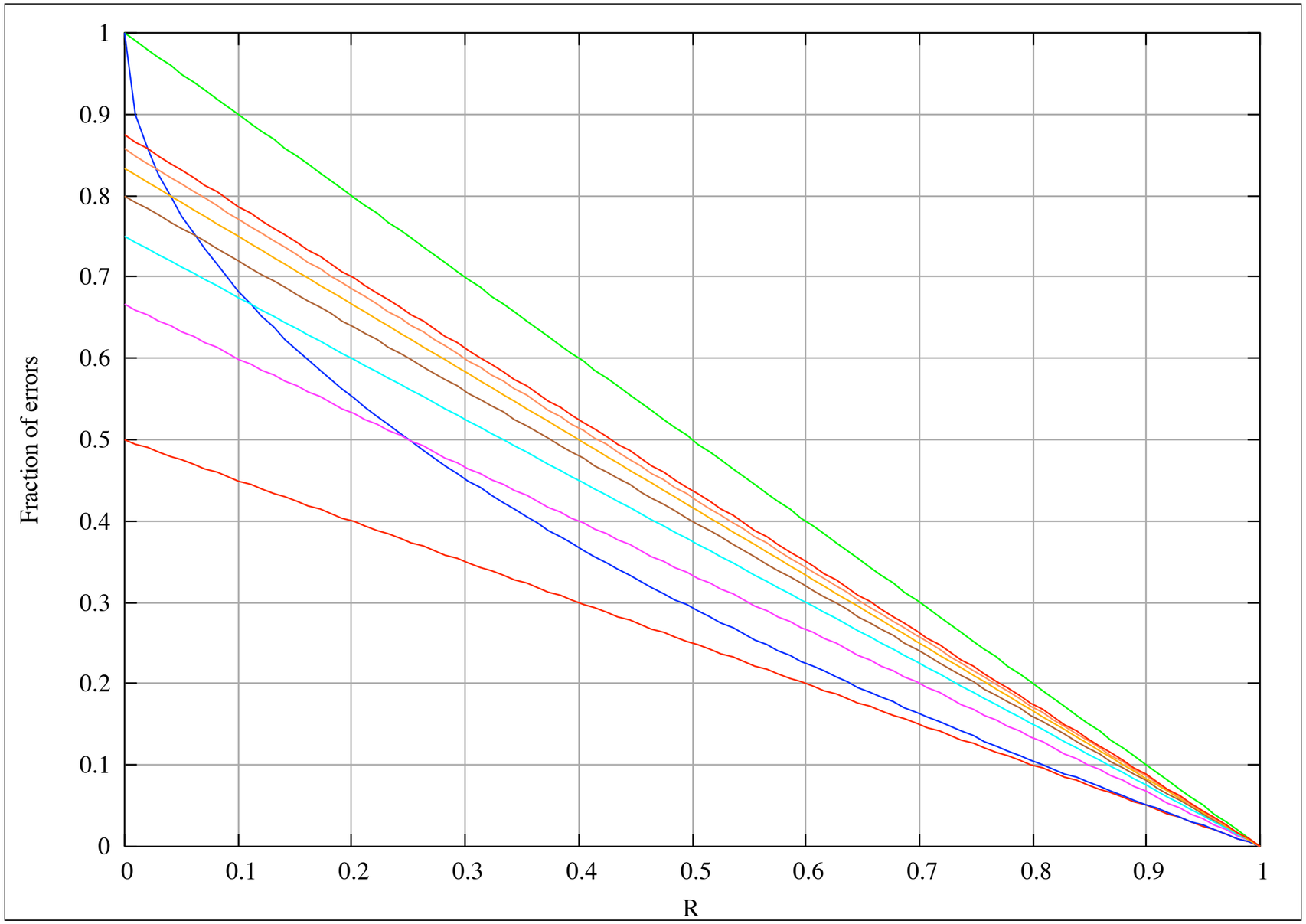

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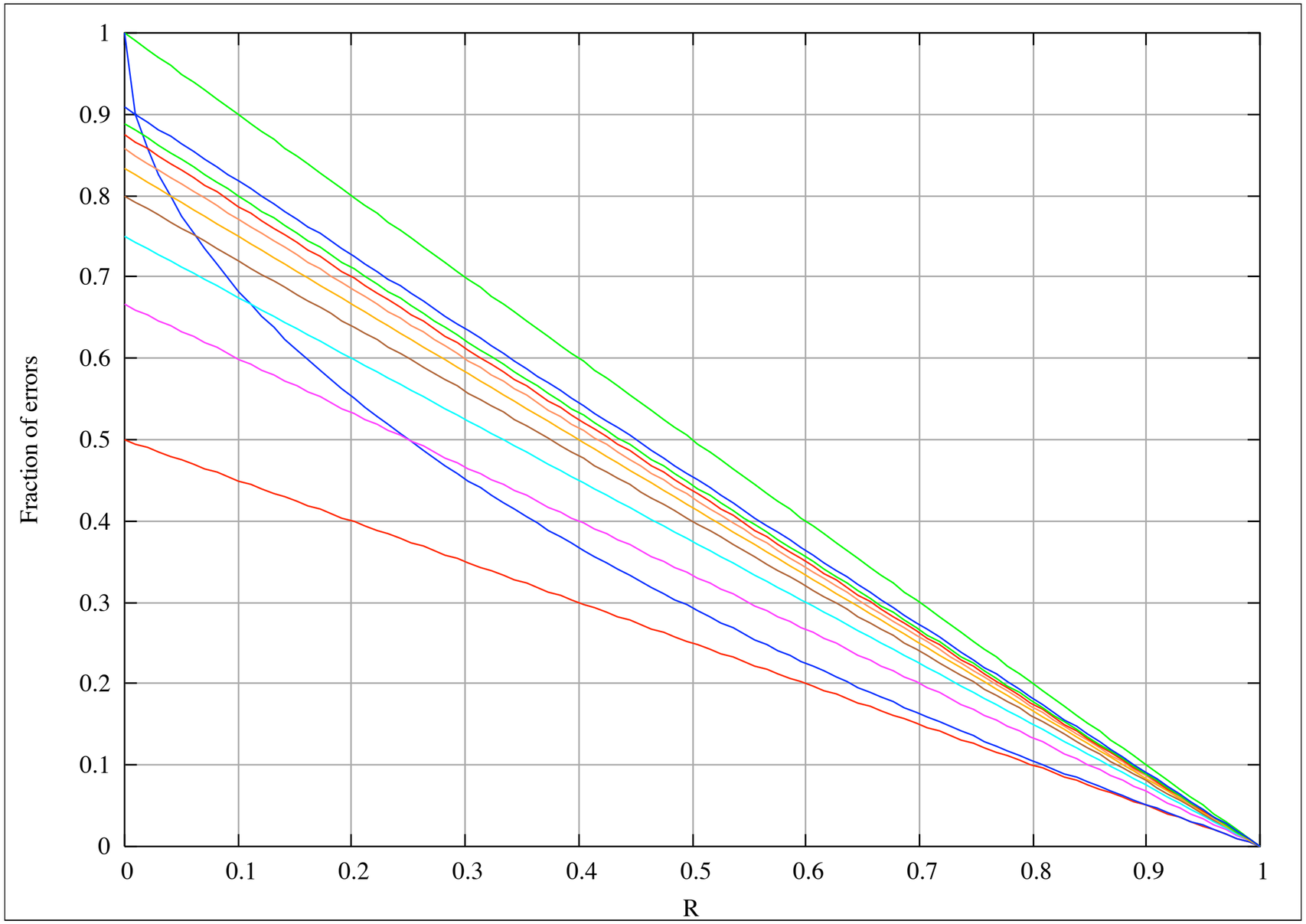

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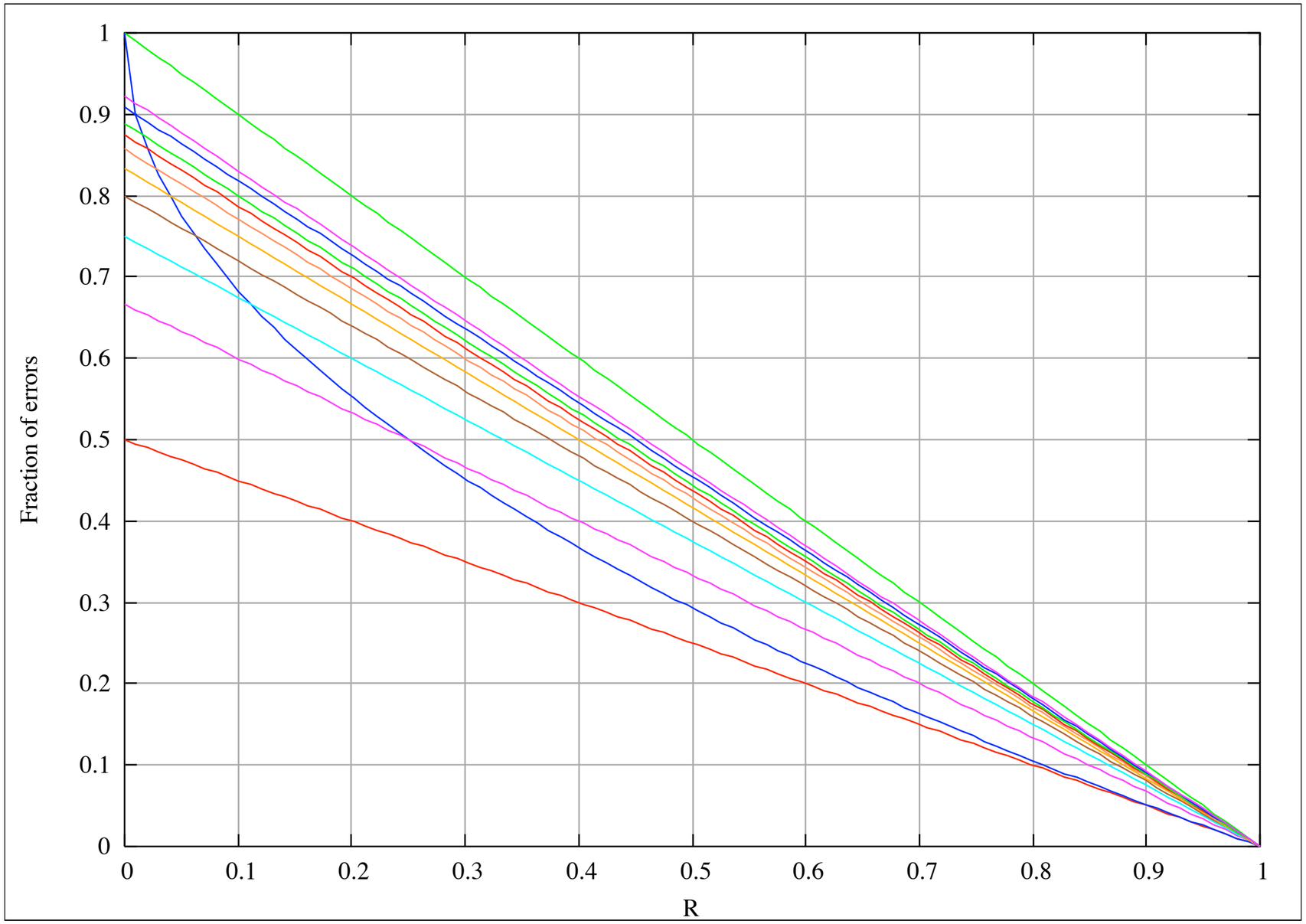

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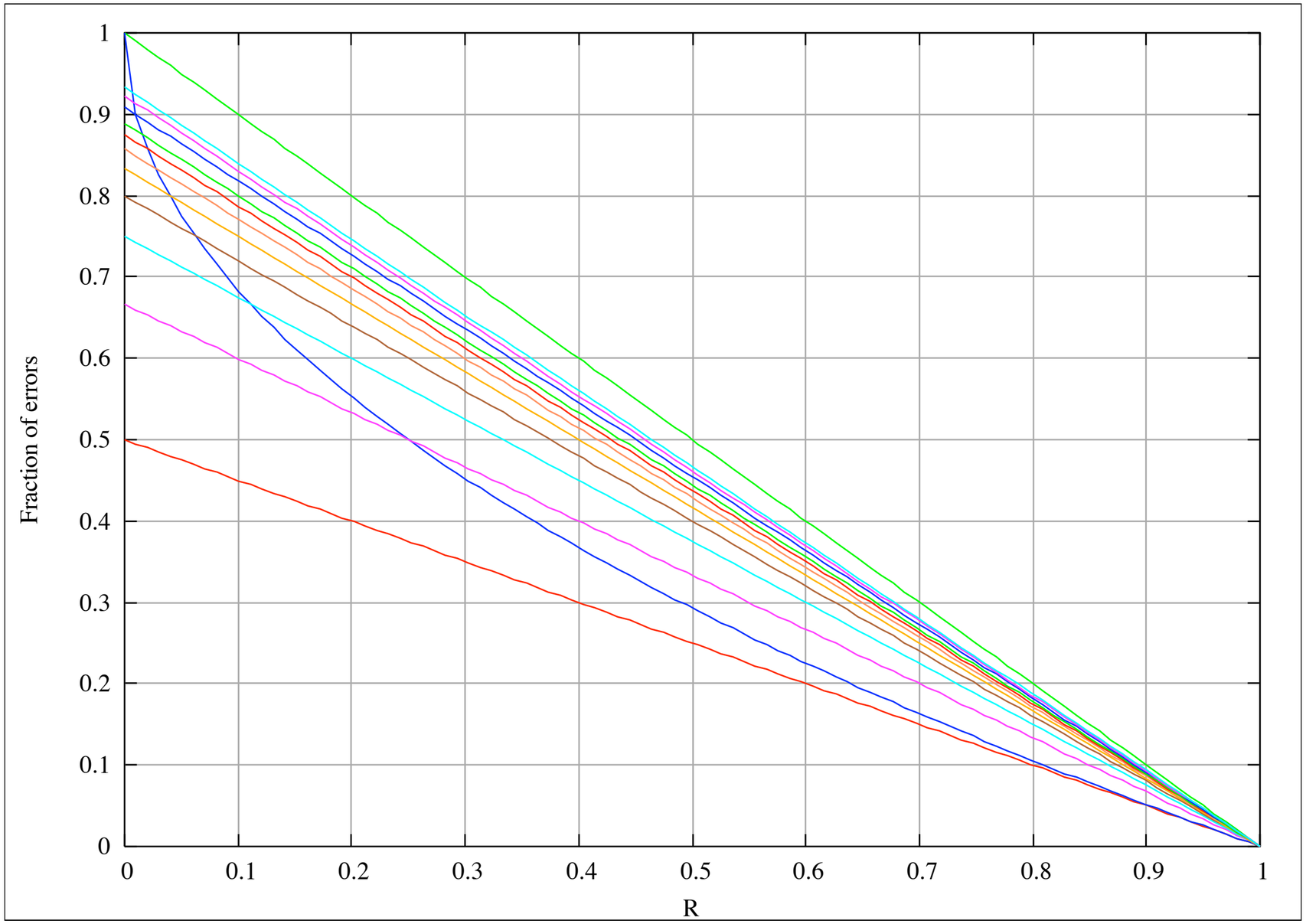

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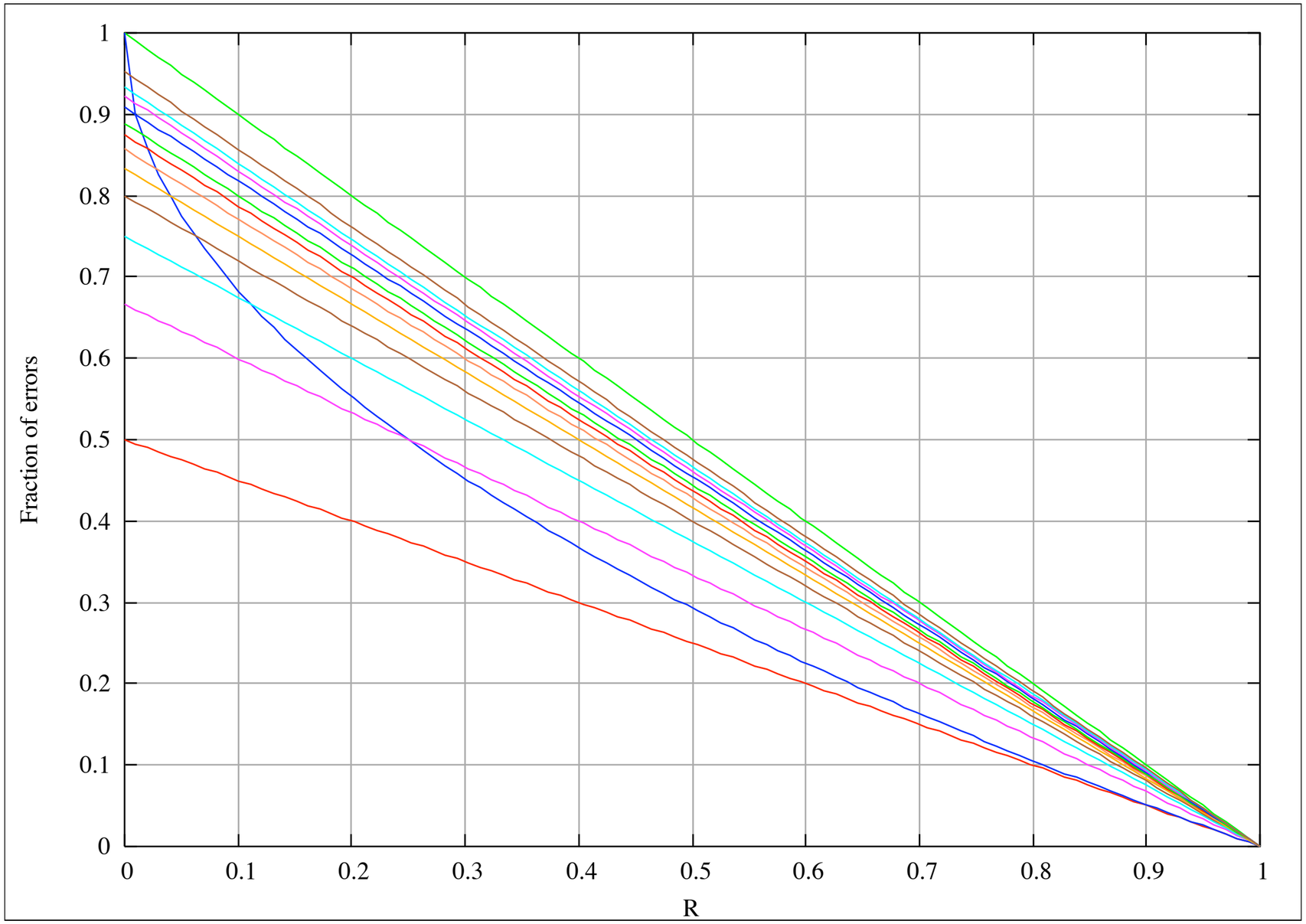

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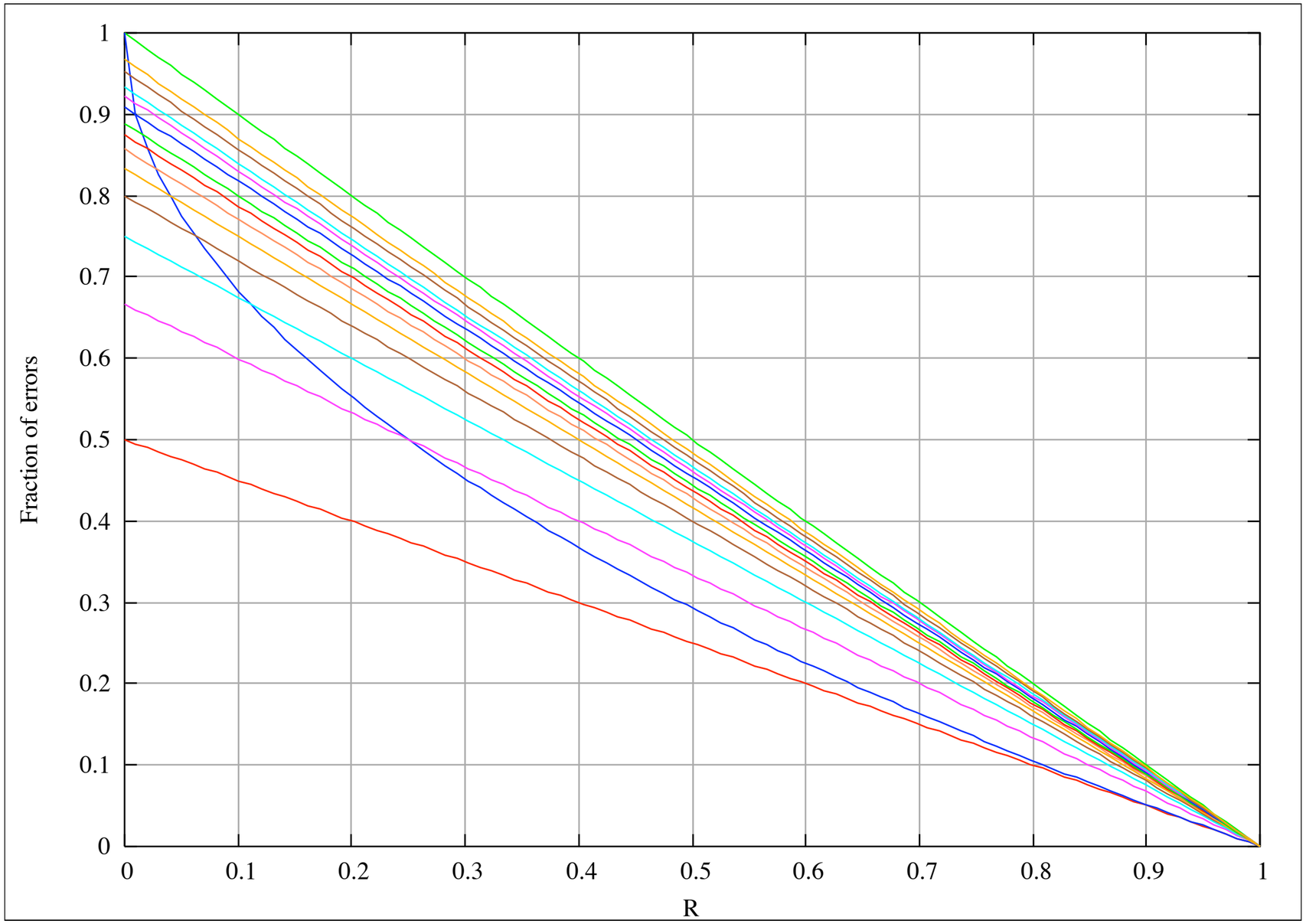

RS-Codes over Large Alphabets

$x_1, \ldots, x_n \in \mathbb{F}_q$ pairwise distinct, $m \geq 1$

$$\begin{array}{rcl} \varphi_m \colon \mathbb{F}_{q^m}[x]_{<k} & \longrightarrow & \mathbb{F}_{q^m}^n \\ f & \longmapsto & (f(x_1), \ldots, f(x_n)) \end{array}$$

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$$\mathbb{F}_{q^m}[x]_{<k} \simeq \left(\mathbb{F}_q[x]_{<k}\right)^m$$

$$\mathbb{F}_{q^m}[x]_{<k} \ni f \leftrightarrow (f_1, \ldots, f_m) \in \left(\mathbb{F}_q[x]_{<k}\right)^m$$

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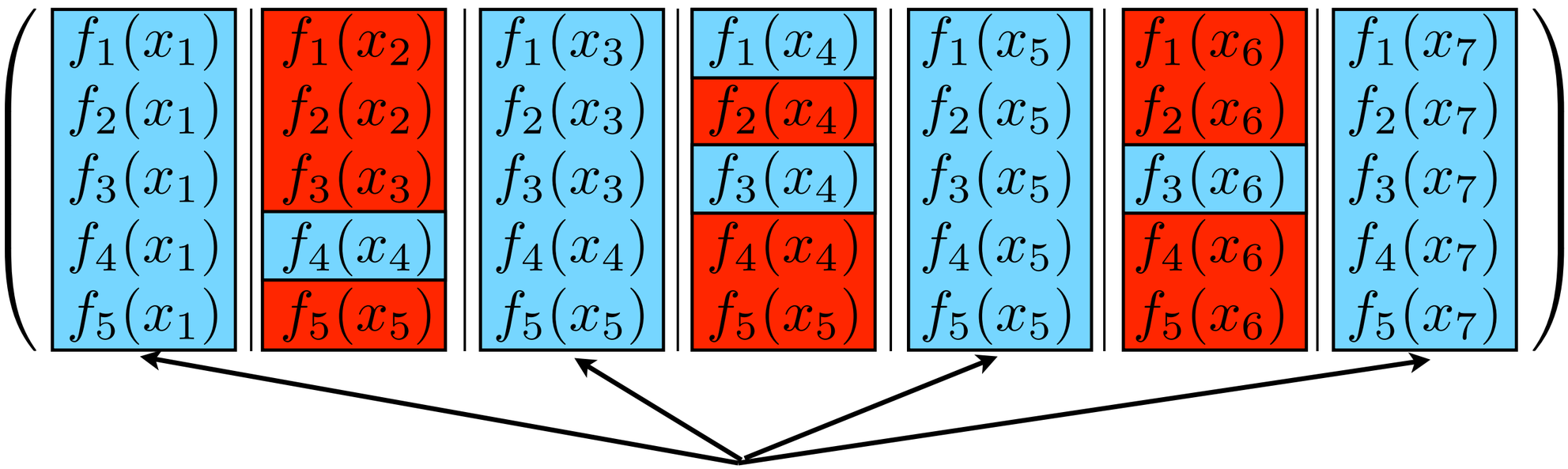
$$\left(\begin{array}{c|c|c|c} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_m(x_1) & f_m(x_2) & \cdots & f_m(x_n) \end{array}\right)$$

Interleaved codeword

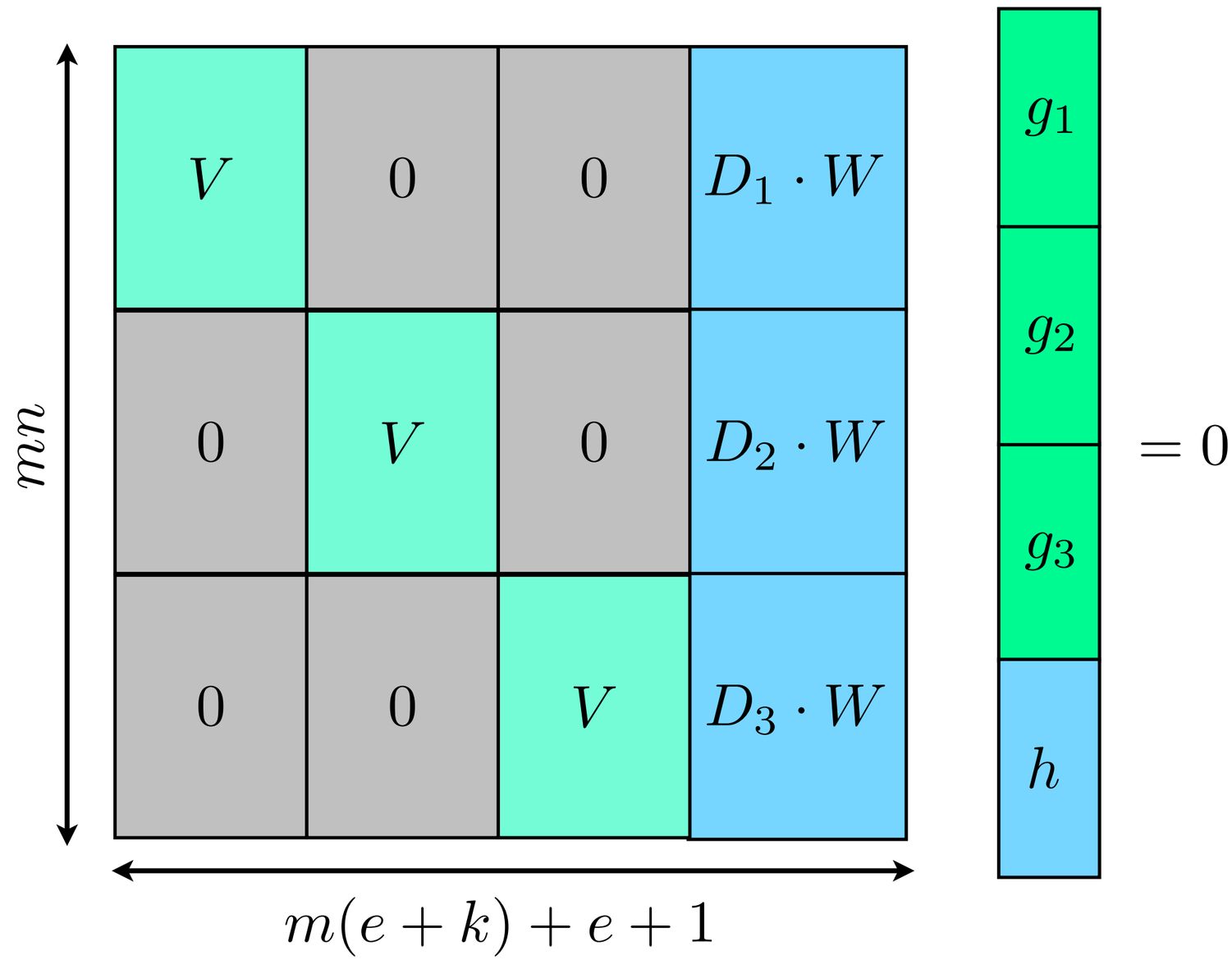
RS-Codes over Large Alphabets



Non-error positions are the same for all the polynomials, so same error locator

$$
\begin{aligned}
g_1(x_i) - y_{i,1}h(x_i) &= 0 \\
g_2(x_i) - y_{i,2}h(x_i) &= 0 \\
g_3(x_i) - y_{i,3}h(x_i) &= 0 \\
g_4(x_i) - y_{i,4}h(x_i) &= 0 \\
g_5(x_i) - y_{i,5}h(x_i) &= 0
\end{aligned}
$$

RS-Codes over Large Alphabets



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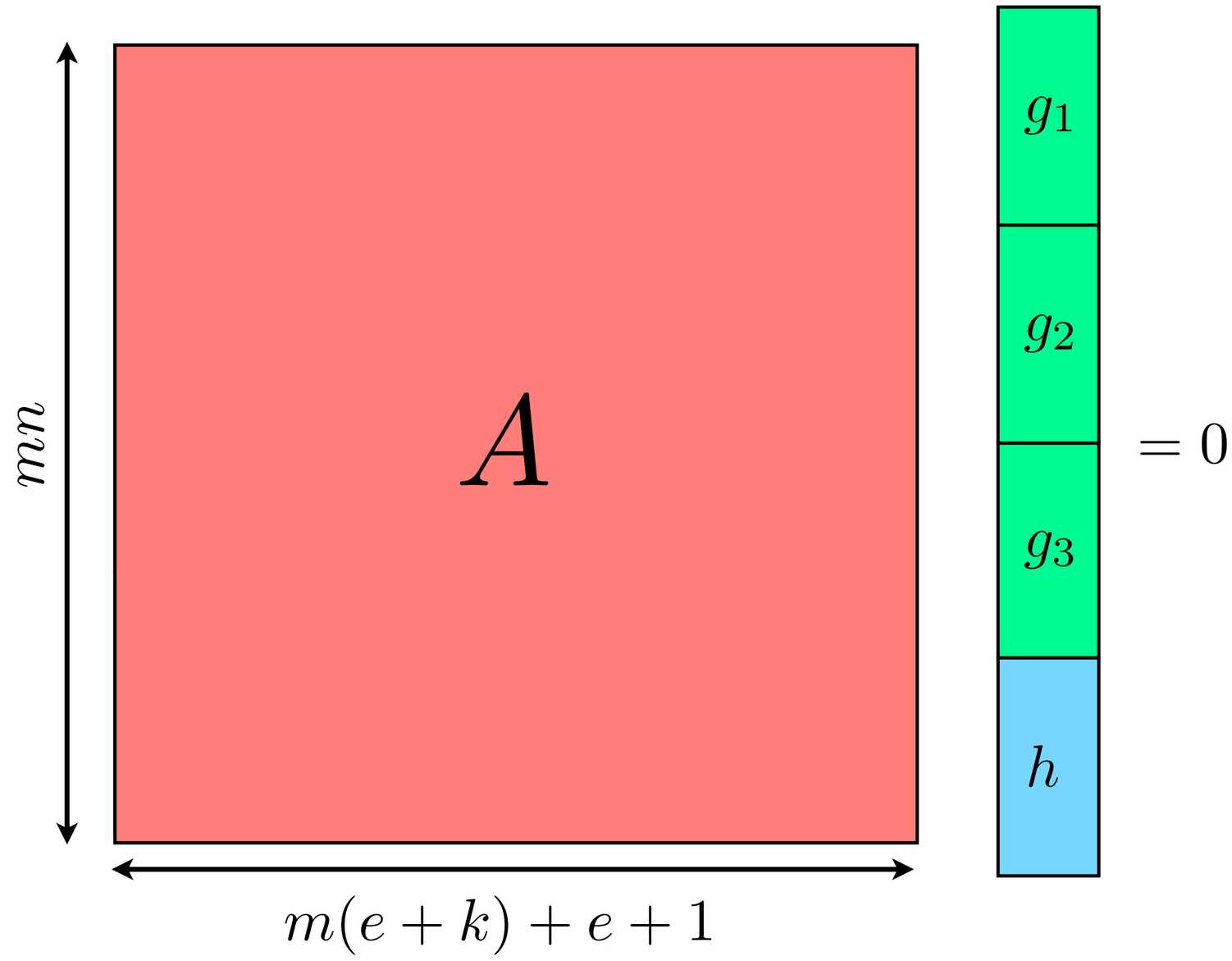
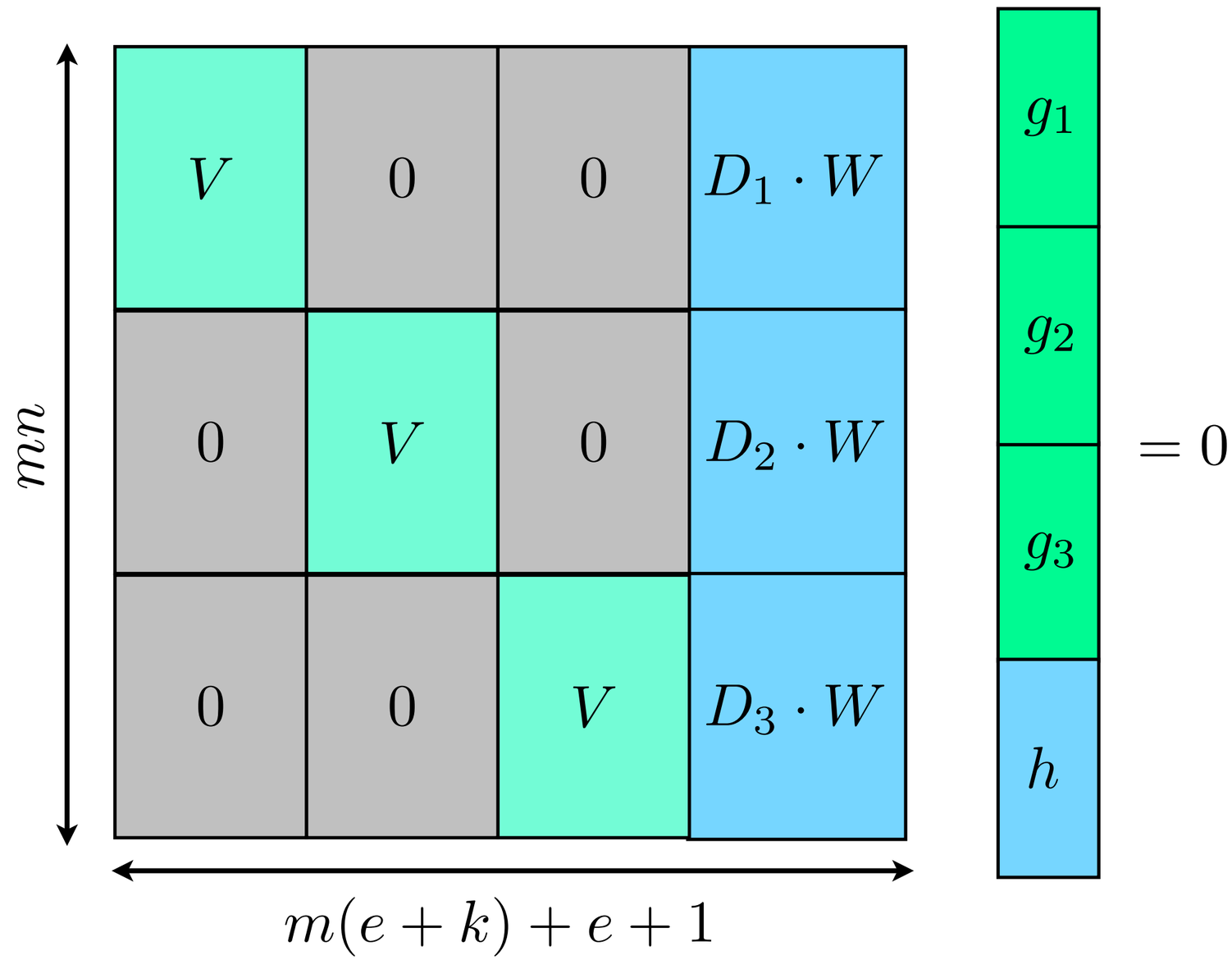


Algorithm

$$A \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ h \end{pmatrix} = 0$$

1. Find element $(g_1|g_2|g_3|h)^\top$ in the right kernel of A.
2. If $g_i/h \notin \mathbb{F}_q[x]_{<k}$ for some i, return error.
3. If not, then set $f_i = g_i/h$.

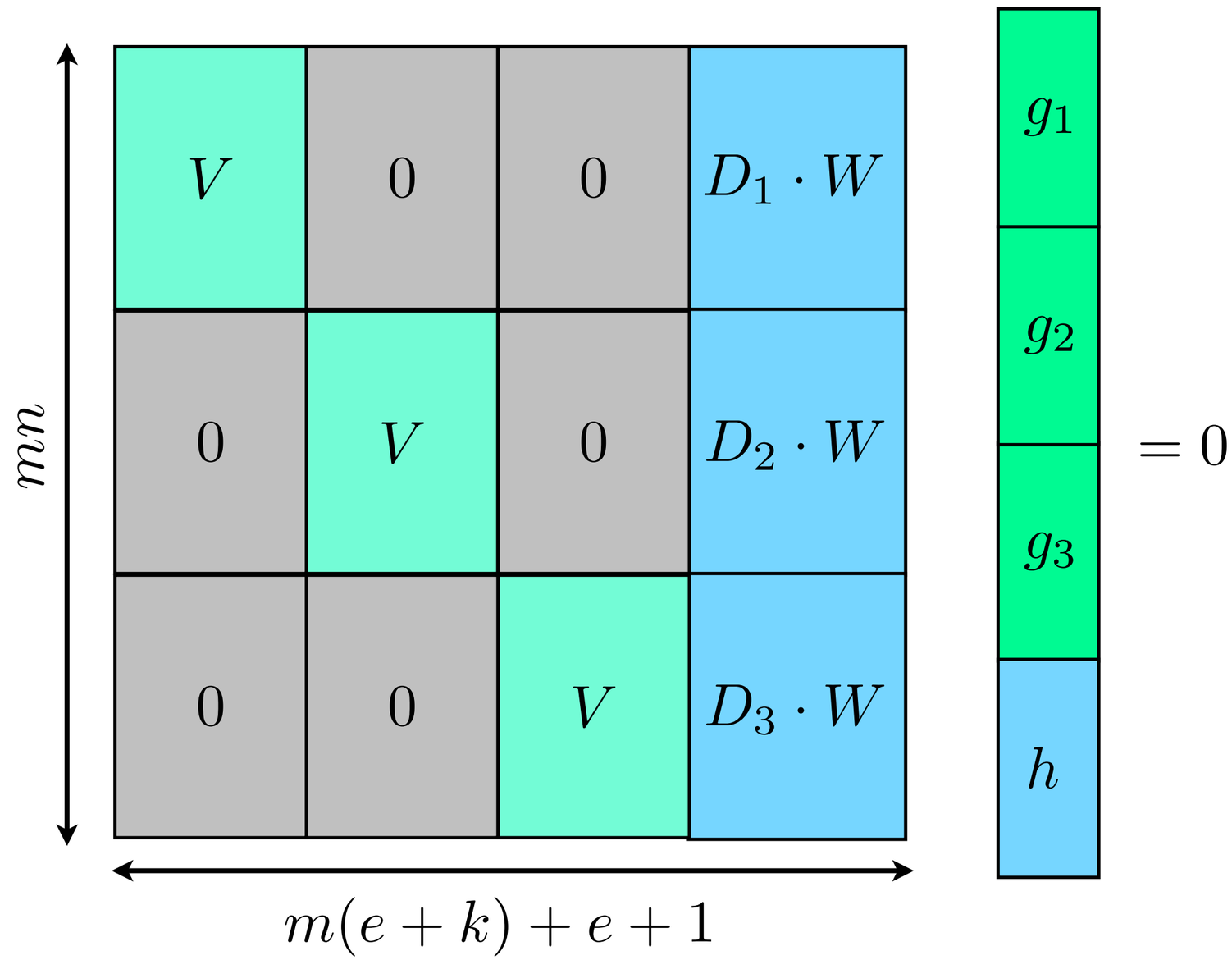
RS-Codes over Large Alphabets



Right kernel is one-dimensional $\Rightarrow$

$$m(e+k)+e+1 = n+1 \Rightarrow \boxed{e = \frac{m}{m+1}(n-k)}$$

Problem



Uniqueness is not guaranteed!

BKY Model

1. Choose $e \leq \frac{m}{m+1}(n-k)$ positions.
2. For each of these positions at least one of the components is in error.
3. The error value is uniformly distributed in $\mathbb{F}_q$.

$$\left(\begin{array}{c|c|c|c|c|c|c} f_1(x_1) & f_1(x_2) & f_1(x_3) & f_1(x_4) & f_1(x_5) & f_1(x_6) & f_1(x_7) \\ f_2(x_1) & f_2(x_2) & f_2(x_3) & f_2(x_4) & f_2(x_5) & f_2(x_6) & f_2(x_7) \\ f_3(x_1) & f_3(x_3) & f_3(x_3) & f_3(x_4) & f_3(x_5) & f_3(x_6) & f_3(x_7) \\ f_4(x_1) & f_4(x_4) & f_4(x_4) & f_4(x_4) & f_4(x_5) & f_4(x_6) & f_4(x_7) \\ f_5(x_1) & f_5(x_5) & f_5(x_5) & f_5(x_5) & f_5(x_5) & f_5(x_6) & f_5(x_7) \end{array}\right)$$

What is the probability of decoding error?

BKY Model

Bleichenbacher et al.: If $e = \frac{m}{m+1}(n-k)$, then the probability of decoding error is $O(n/q)$.

Brown et al.: If $e = \frac{m}{m+1}(n-k)$, then the probability of decoding error is $O(1/q)$.

This talk: Roughly, if $e = \frac{m}{m+1}(n-k) - \epsilon n$, then the error probability of the decoder is $O(q^{-m\epsilon n})$.

BKY Model

Suppose that

$$e \leq \frac{\beta m}{\beta m + 1}(n-k) - \frac{c}{\beta m + 1},$$

for some $c > 0$, and where $\beta = \frac{\ln(q^m - 1)}{\ln(q^m)}$. Then we have:

(1) If $e + t < n - k$, then the error probability of the decoder is zero.

(2) In general the error probability of the decoder is at most $\frac{q}{q-1} \cdot q^{-c}$.

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$$\beta \simeq 1$$

$$\text{Roughly: } e \leq \frac{m}{m+1}(n-k) - \epsilon n \Rightarrow \text{Error probability } \simeq q^{-m\epsilon n}$$

Method of Proof

$$A \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ h \end{pmatrix} = 0$$

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Method of Proof

$$A \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ h \end{pmatrix} = 0$$

Need to show that right kernel of A is one-dimensional, w.h.p.

Calculate expectation of size of right kernel, and apply Markov.

Complexity

$$A \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ h \end{pmatrix} = 0$$

$$O((mn)^3 \log^2(q))$$

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$$O((mn) \log^2(mn) \log^2(q))?$$

AG-Codes

For a variety of reasons, we are interested in constructing codes for which q and m are fixed, but the length goes to infinity.

Not possible for RS-codes: the length of the code is no more than q+1.

AG-codes offer a solution! They are constructed by

(a) Realizing the RS-codes are related to the projective line, and
(b) replacing the projective line with an algebraic curve.

The algorithm described has to be properly generalized to AG-codes, and the new algorithm has to be analyzed in this setting.

AG-Codes

In this case the upper bound on the number of correctable errors is

$$\frac{\beta m}{\beta m + 1}(n - k) - 2g$$

where g is the genus of the curve. The probabilistic statements remain with respect to this bound.

Open Questions

1. Can we get better error-correction capability with exponential error bounds and same or better running times?

2. Is it possible to improve the running time of the algorithms to

$$O(mn^{1+\epsilon}\log^{1+\epsilon}(q))?$$

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2. Is it possible to improve the running time of the algorithms to

$$O(mn^{1+\epsilon}\log^{1+\epsilon}(q))?$$

3. Is it possible to make the algorithms "practical?"

4. Are there other applications of "Interleaved codes?"

~~Open~~ Questions

Thank you.

Diamantina, August 2006