Capacity Achieving Sequences for the Erasure Channel

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Outline

This talk will provide:

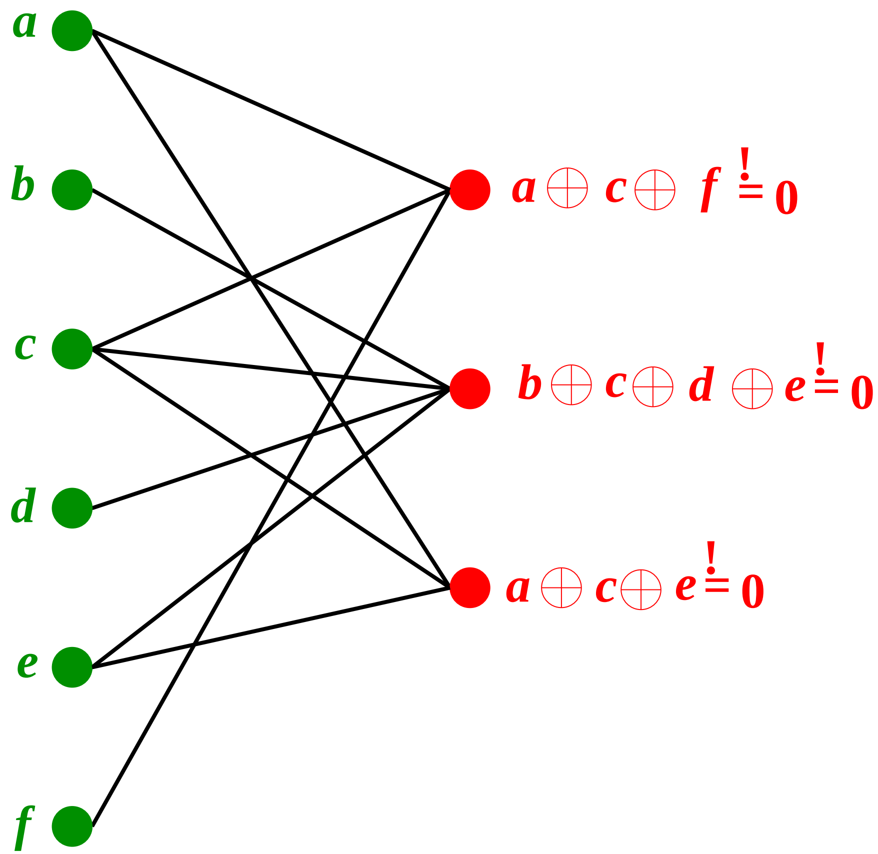
- Partial description of capacity achieving sequences for the erasure channel.

- Notes on the convergence speed.

- Notes on the asymptotic fraction of degree 2 nodes in terms of convergence speed.

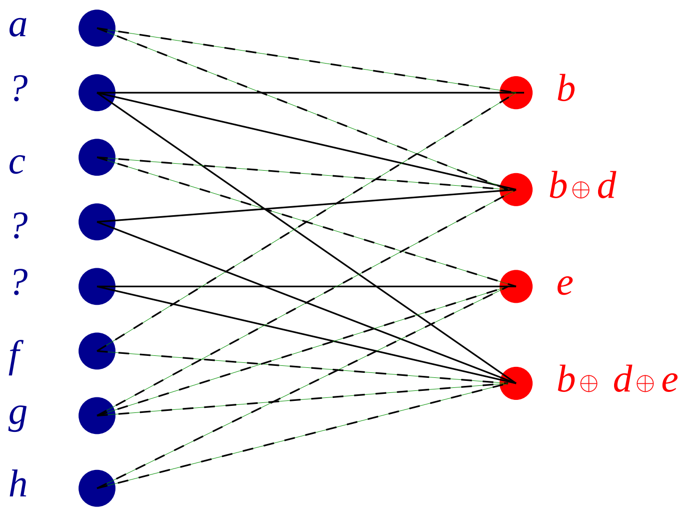
- Reasons why linear programming methods behave as they do!

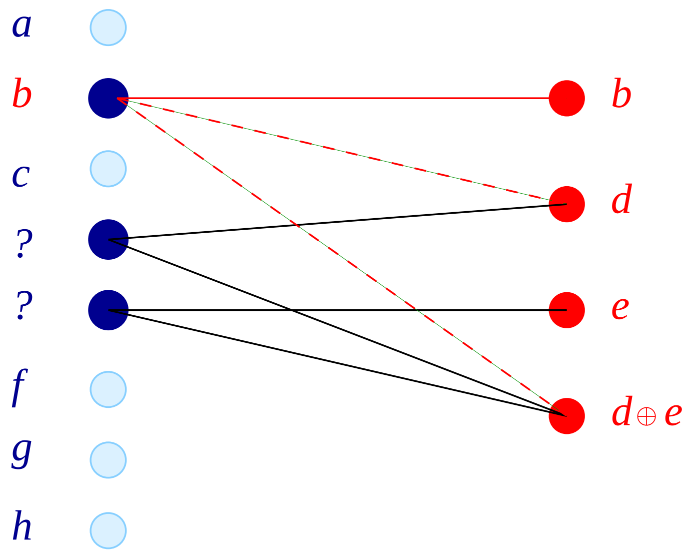
LDPC Codes



a
b
c
d
e
f
$a \oplus c \oplus f \stackrel{!}{=} 0$
$b \oplus c \oplus d \oplus e \stackrel{!}{=} 0$
$a \oplus c \oplus e \stackrel{!}{=} 0$

Erasure Decoding





Successful Decoding

$$p_0\lambda(1-\rho(1-x)) < x \qquad \text{for } 0 < x < p_0.$$

Design degree distributions so that maximal $p_0 \sim 1 - R$ and average degree of graph small. ($R =$ rate.)

Capacity Achieving Sequences

Want sequences for which maximal p_0 can be arbitrarily close to $1 - R$.

- Tornado sequence—Luby et al. 1997: $\lambda(x) = \frac{1}{H(D)} \sum_{i=1}^{D} \frac{x^i}{i}$, $\rho(x) = \mathrm{e}^{\alpha(x-1)}$.

- Right-regular sequence—Shokrollahi 1999: $\rho(x) = x^d$.

- Other sequences?

Some Function Spaces

$$\mathcal{P} := \left\{ f(x) = \sum_{1}^{\infty} f_k x^k, \quad x \in [0,1] \;\middle|\; f_k \geq 0, \quad f(1) = 1 \right\},$$

$$\mathcal{A} := \{ f \in \mathcal{P} \mid \mathcal{T} f \in \mathcal{P} \},$$

where

$$\mathcal{T} f(x) := 1 - f^{-1}(1 - x).$$

Functions in $\mathcal{A}$ can be used to design degree distributions.

Properties

- $\mathcal{T}(f \circ g) = \mathcal{T}g \circ \mathcal{T}f$, so $\mathcal{A}$ is closed under $\circ$ since $\mathcal{P}$ is. $((f \circ g)(x) := f(g(x))$.

- $f \in \mathcal{A}$, then $\frac{f(ax+b)-f(b)}{f(a+b)-f(b)} \in \mathcal{A}$ for any $0 < a \leq a+b \leq 1$.

Can construct new capacity achieving sequences from old ones:

- $f(x) = \frac{e^{ax}-1}{e^{a}-1}, a > 0$ (Tornado): $f \circ f$,

- $f(x) = x^n$ (right-regular): $x^{2n} + x^n$,

- $f(x) = \frac{(1-b)x}{1-bx}$, $0 < b < 1$.

Convergence Speed

How fast (in terms of their average degrees) do these graphs converge to capacity?

P largest fraction of tolerable erasures, A average degree.

Shokrollahi 1999: if $P = 1 - R - \epsilon$, then

$$\epsilon \geq R^A, \text{ i.e., } A = \Omega(\log(1/\epsilon))$$

for fixed R;

Tornado codes and right regular codes are optimal in the *logarithmic sense*.

Convergence Speed

Measure: $\epsilon \geq R^A$,

$$\mu := \frac{\log\left(\frac{1}{\epsilon}\right)}{A \log\left(\frac{1}{R}\right)}, \qquad \Delta := \frac{\epsilon}{R^A}$$

Tornado sequence:

$$\mu \to \frac{\log(1/R)}{1-R}, \qquad \Delta \to \infty.$$

Right-regular sequence

$$\mu \to 1 \qquad \Delta \to e^{\gamma} = 1.7810...$$

Convergence Speed

No other sequence known for which Δ converges to a smaller number.

Conjecture: The right-regular sequence is optimal!

Gives reason to why linear programming approaches favor one degree on the right.

Degree 2 Nodes

Can we get the fraction of degree 2 nodes asymptotically below $1 - R$?
(Has consequences for the encoding/decoding.)

Take $\phi \in \mathcal{A}$ and consider the sequence with right distribution $\phi(x^n)$.

μ corresponding constant for this sequence.

Λ_2^n is fraction of degree 2 nodes of corresponding left hand side.

$$\lim_{n\to\infty} \Lambda_2^n = \frac{\mu}{2\phi'(1)}.$$

Degree 2 Nodes

Fraction can be made less than $1 - R$, while still capacity achieving.

The smaller this fraction, the large corresponding μ of the sequence. Explains LP-results!

Is this result true in general?

Are there capacity achieving codes with $\Lambda_2 = 0$ for almost all elements in the sequence?

Does not contradict the flatness condition