4-Antenna Space-Time Codes from Representations of $SU(2)$

Amin Shokrollahi

Outline

In this talk we show how good 4-antenna differential unitary space-time codes can be constructed using restricted 3-dimensional spherical codes and the irreducible 4-dimensional representation of the Lie group $SU(2)$.

Basic Code Design Problem: Given an integer M, find a large subset $\mathcal{S} \subset U(M)$ for which the diversity distance

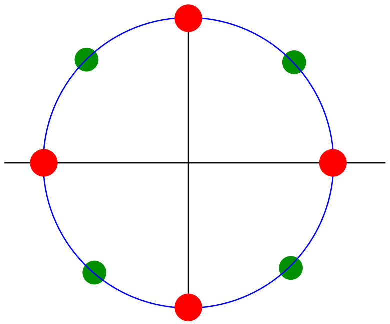
$$\zeta(\mathcal{S}) := \frac{1}{2} \min_{S,R \in \mathcal{S}, S \neq R} |\det(S - R)|^{1/M}$$

is as large as possible. (Hochwald-Sweldens, Hughes)

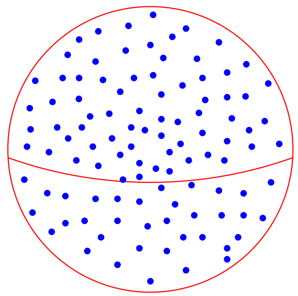
Special Cases for M

Main Problem: Diversity distance is not a distance!

$M = 1$: PSK constellations



$M = 2$: Diversity distance is distance if restricted to $SU(2)$. Good codes correspond to good spherical codes on the 3-dimensional sphere.



Larger M

- Together with Hassibi, Hochwald, and Sweldens we classified all finite groups with positive diversity distance.

- What about infinite groups?

- Hassibi and Khorrami: Only Lie groups with positive diversity distance are the obvious ones, i.e., $U(1)$ and $SU(2)$.

- What about subsets of Lie groups?

This talk concentrates on subsets of $SU(2)$ under embedding into $SU(4)$.

Irreducible Representations of Lie Groups

Need unitary and finite dimensional representations of Lie groups.

Theorem of Peter-Weyl: All irreducible representations of compact Lie groups are finite dimensional and unitary!

First example: 4-dimensional representation of $SU(2)$.

4-dimensional Representation of $SU(2)$

Start from obvious definition and use invariant integration:

$$R\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} a^3 & \sqrt{3}a^2b & \sqrt{3}ab^2 & b^3 \\ -\sqrt{3}a^2\overline{b} & a(|a|^2 - 2|b|^2) & b(2|a|^2 - |b|^2) & \sqrt{3}\overline{a}b^2 \\ \sqrt{3}a\overline{b}^2 & \overline{b}(|b|^2 - 2|a|^2) & \overline{a}(2|b|^2 - |a|^2) & \sqrt{3}b\overline{a}^2 \\ -\overline{b}^3 & \sqrt{3}\ \overline{b}^2a & -\sqrt{3}\ \overline{b}\overline{a}^2 & \overline{a}^3 \end{pmatrix}$$

Eigenvalues

The eigenvalues of $R(A)$ are $\lambda, \overline{\lambda}, \lambda^3, \overline{\lambda}^3$ if those of A are $\lambda, \overline{\lambda}$. (Comes from the obvious definition of the representation.)

Need set $\mathcal{A}$ of matrices in $SU(2)$ such that for all $A, B \in \mathcal{A}$ the eigenvalues of AB^* are away from third roots of unity.

Elements of $SU(2)$ correspond to points on the 3-dimensional sphere in the 4-dimensional space.

$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mapsto (\mathrm{Re}(a), \mathrm{Im}(a), \mathrm{Re}(b), \mathrm{Im}(b)) \in \mathbb{R}^4.$$

Restricted Spherical Codes

AB^* has a third root of unity as eigenvalue iff the angle between the associated points on the sphere is is 120 degrees.

Need spherical codes in which no two points have such an angle.

How do we find them?

Restricted Spherical Codes

Assumption: Very good spherical codes are well-spread.

Want to find subset $\mathcal{S}$ of 3-dimensional sphere of large volume in which no two points have an angle of 120 degrees.

Restrict a good spherical code in $\mathbb{S}^3$ to this subset to obtain the restricted code.

Restricted Spherical Codes

$$\Sigma_{a,b,c} := \Big\{(\cos(\phi)\cos(\psi)\cos(\theta), \sin(\phi)\cos(\psi)\cos(\theta), \sin(\psi)\cos(\theta), \sin(\theta)) \mid$$
$$0 \le \phi \le a, -b \le \psi \le \frac{\pi}{2}, -c \le \theta \le \frac{\pi}{2}\Big\}.$$

Minimal value of $\langle x, y \rangle$ for $x, y \in \Sigma_{a,b}$ is

$$-\sqrt{\sin(c)^2 + (\sin(b)^2 + \cos(a)^2 \cos(b)^2)\cos(c)^2},$$

Volume of $\Sigma_{a,b,c}$ is

$$\mathrm{vol}(\Sigma_{a,b,c}) = a(1 + \sin(b))\left(\frac{\pi}{4} + \frac{1}{2}\cos(c)\sin(c) + \frac{1}{2}c\right).$$

Optimization

$$\Sigma \;:=\; \Big\{(\cos(\phi)\cos(\psi)\cos(\theta), \sin(\phi)\cos(\psi)\cos(\theta), \sin(\psi)\cos(\theta), \sin(\theta)) \mid$$
$$0 \le \phi \le 1.8174, -0.319 \le \psi \le \frac{\pi}{2}, -0.3467 \le \theta \le \frac{\pi}{2}\}.$$

Then $\mathrm{vol}(\Sigma)/\mathrm{vol}(\mathbb{S}^3) > 0.135$ and any two points in Σ have a scalar product which is at least -0.4985.

Uses Differential Evolution.

Further Remarks

- Decoding?
- Simulation results?
- Other compact Lie Groups?