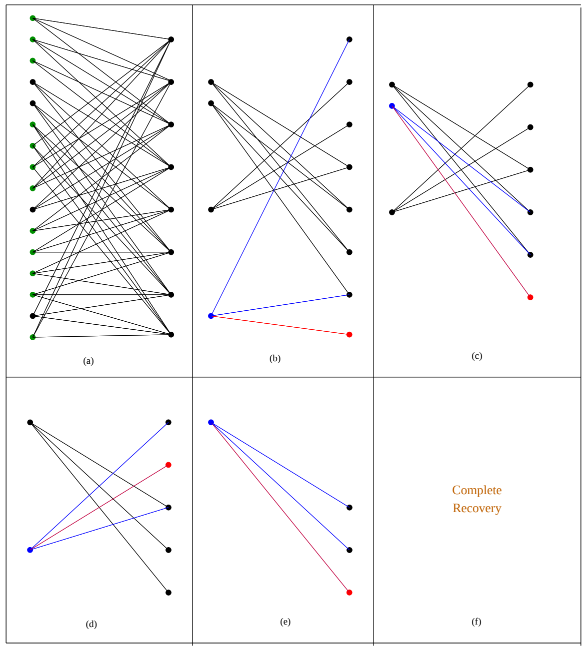
Low-Density Codes and the Erasure Channel



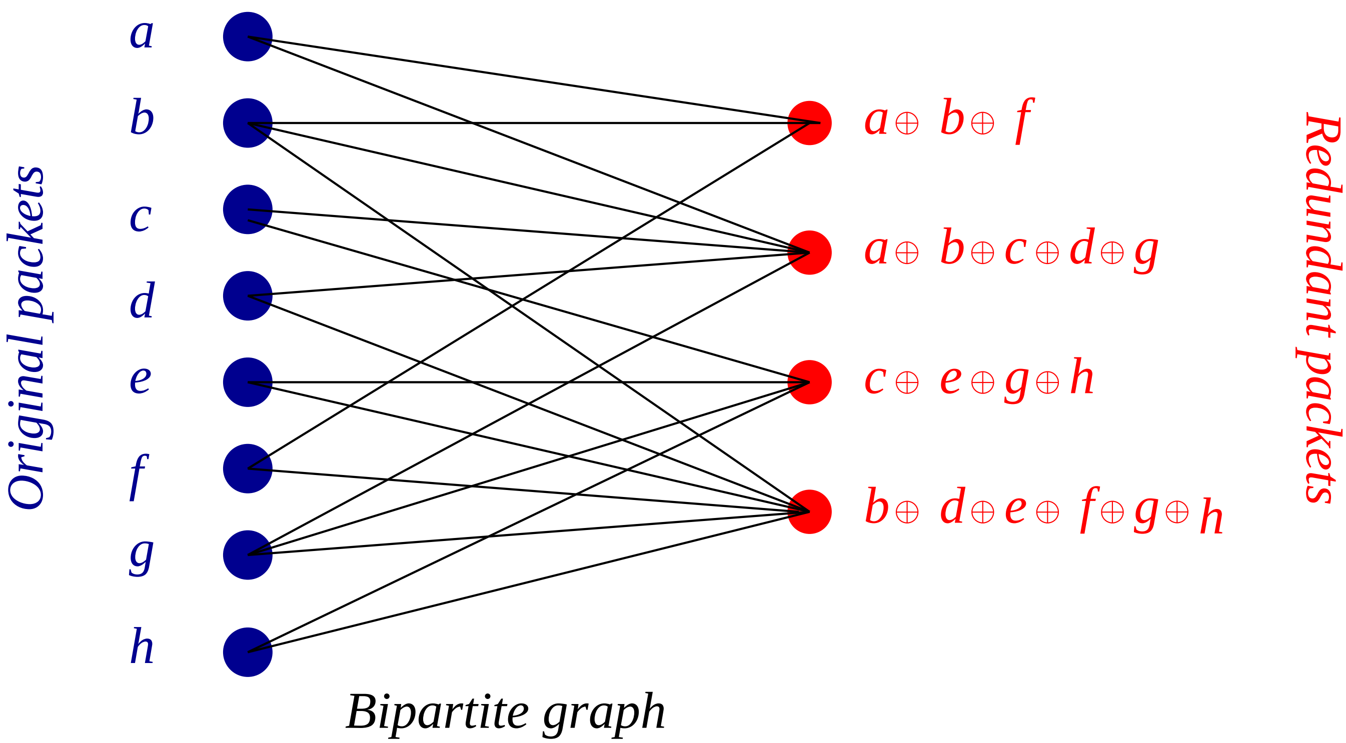
M. Amin Shokrollahi

Bell Laboratories

Lucent Technologies
Bell Labs Innovations

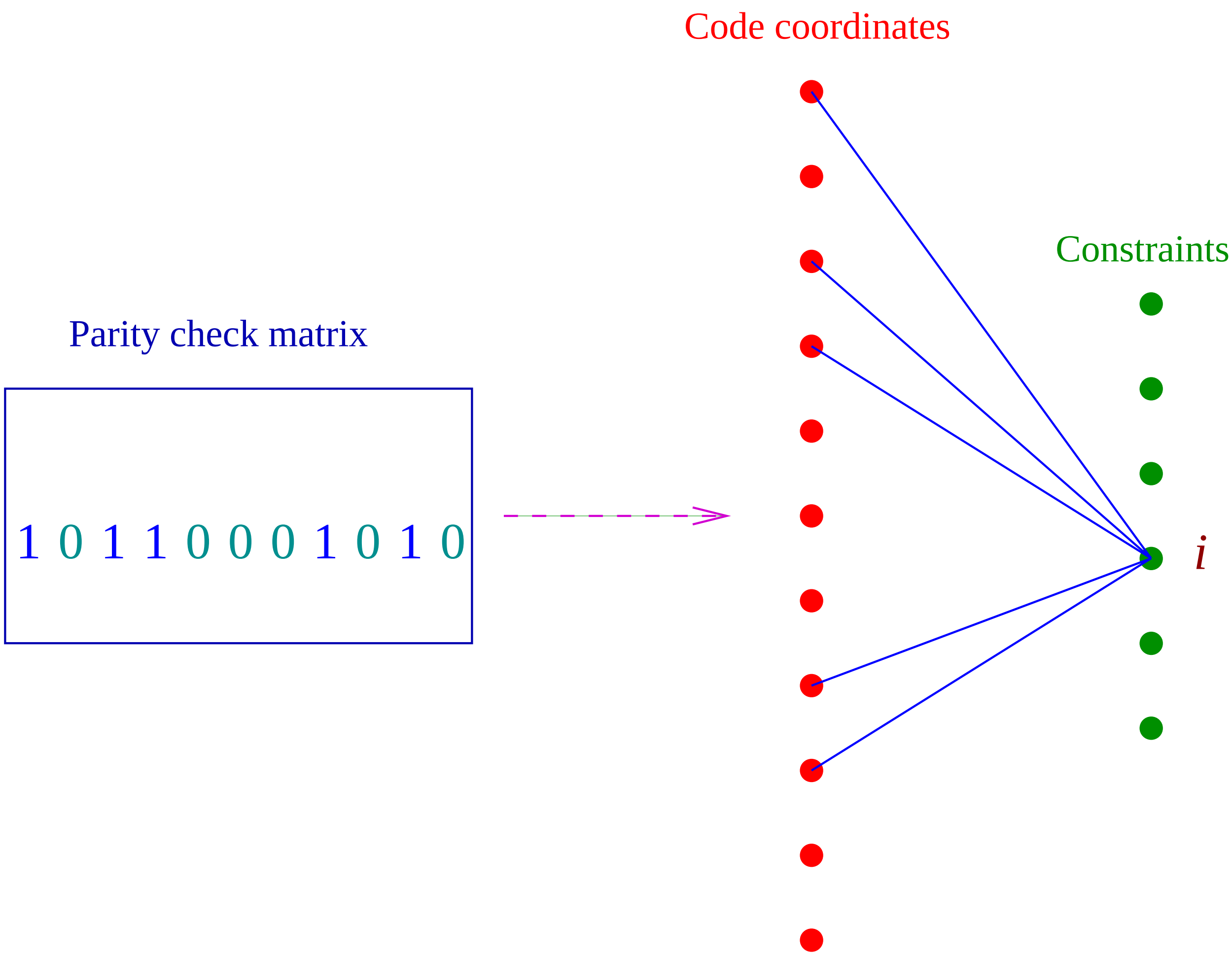
Encoding with Bipartite Graphs

Packets = Bits

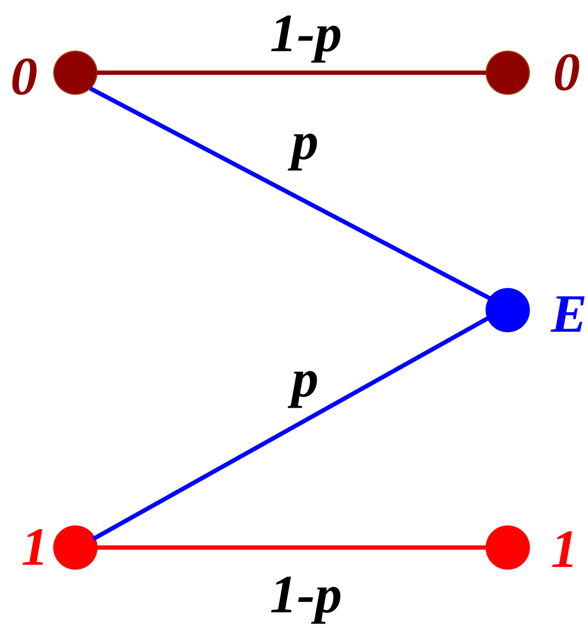


Encoding time is proportional to number of edges in graph.

Different Perspective

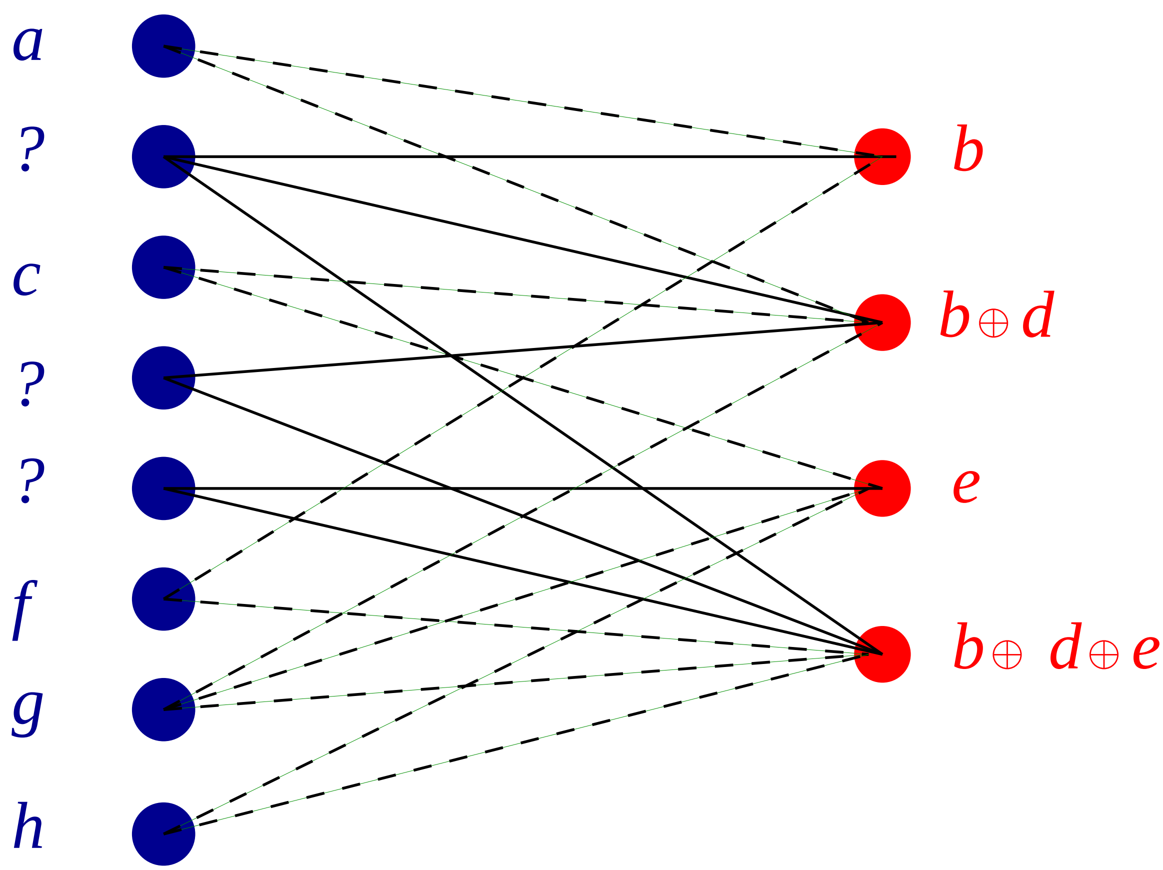


Erasure Channel



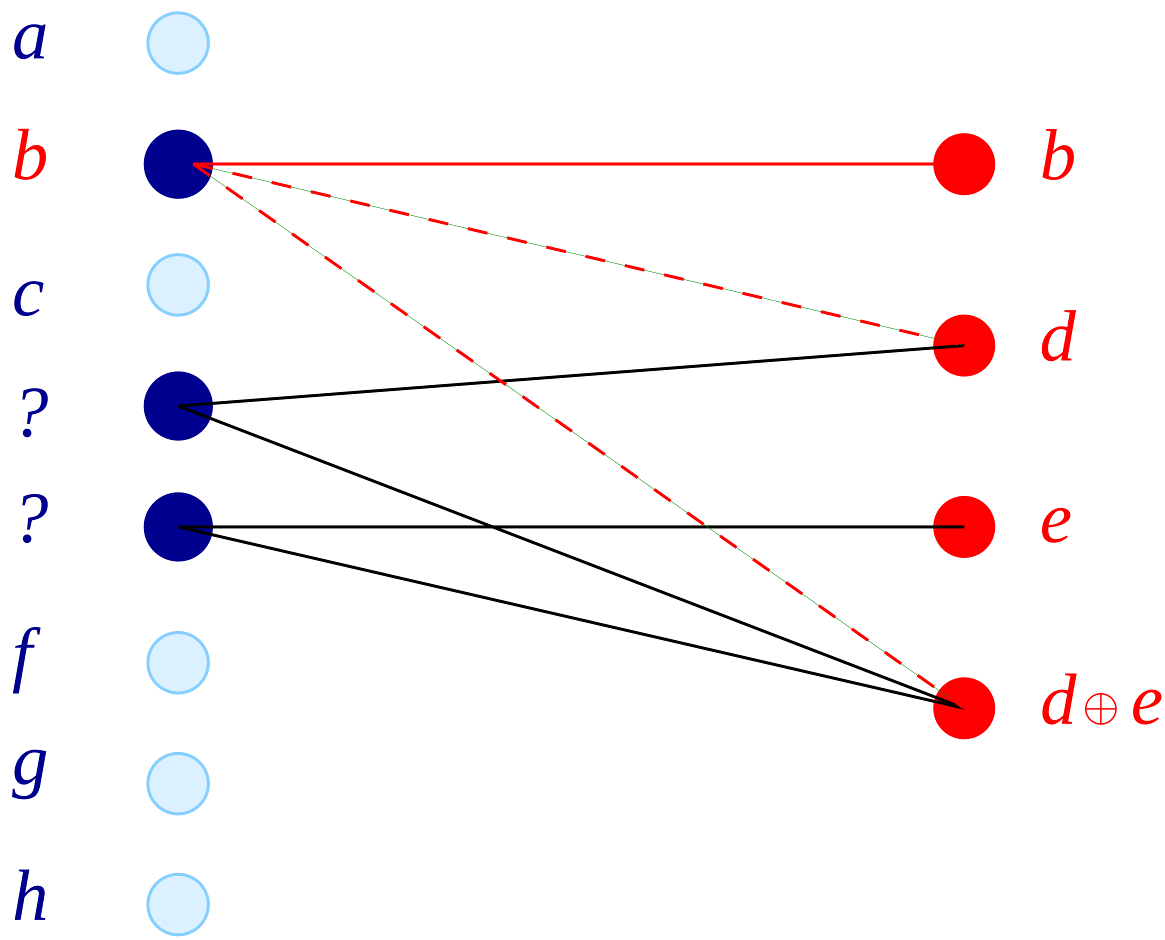
Decoding

Stage 1: Direct Recovery



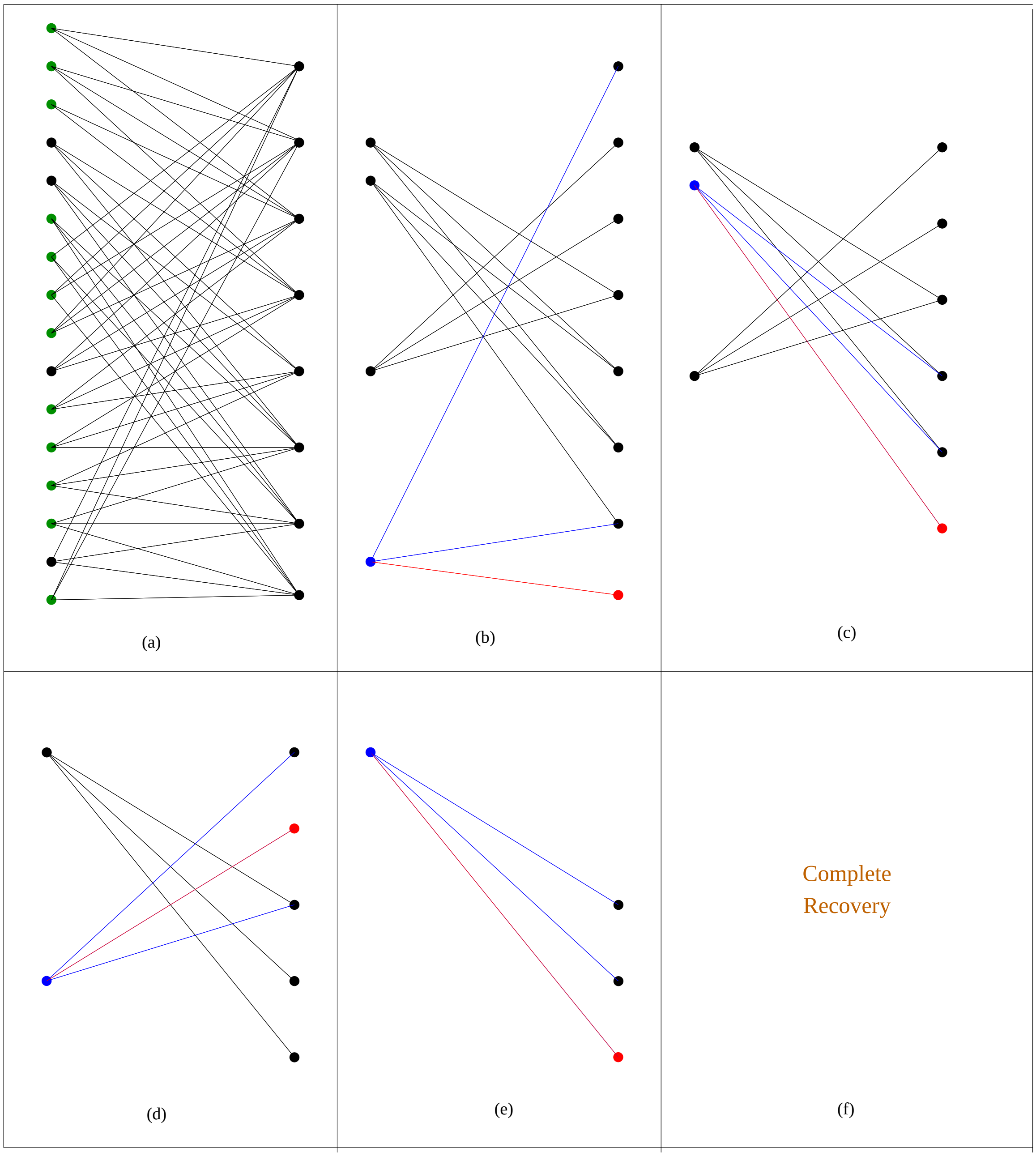
Decoding

Stage 2: Substitution Recovery



Decoding time is proportional to number of edges in graph.

Example



The Problem

Have fast encoding and decoding algorithm, if graph is sparse.

Want to design codes (=graphs) that perform good with respect to these algorithms.

How?

Experiments

Choose regular graphs.

A $(3, 6)$-graph recovers from 42.9% erasures.

A $(4, 8)$-graph recovers from 38.3% erasures.

A $(5, 10)$-graph recovers from 34.1% erasures.

What are these numbers?

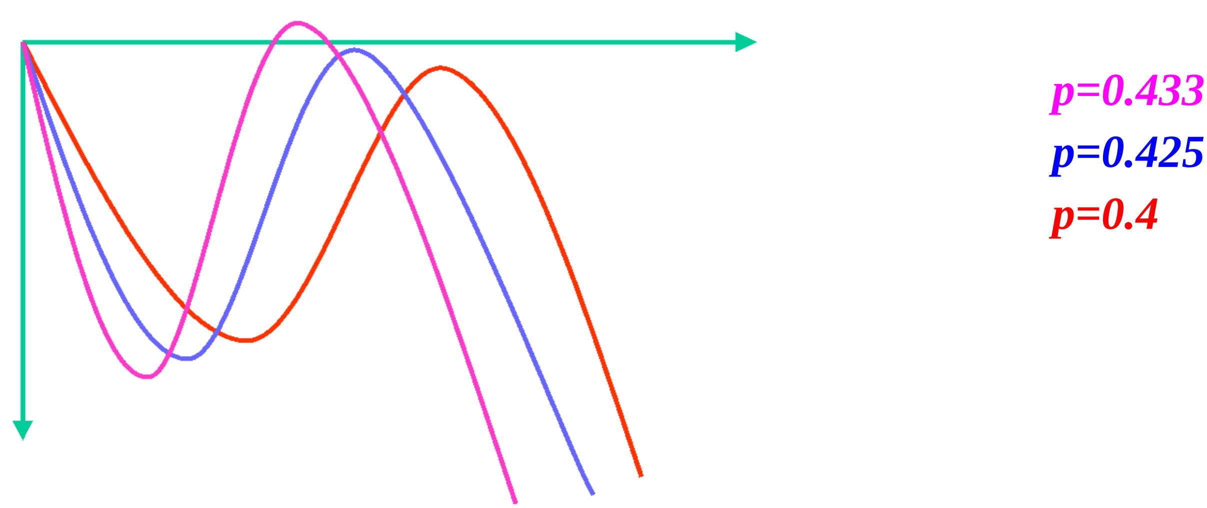
Revelation

Theorem. A random (k, d)-graph recovers from a p-fraction of erasures with high probability iff

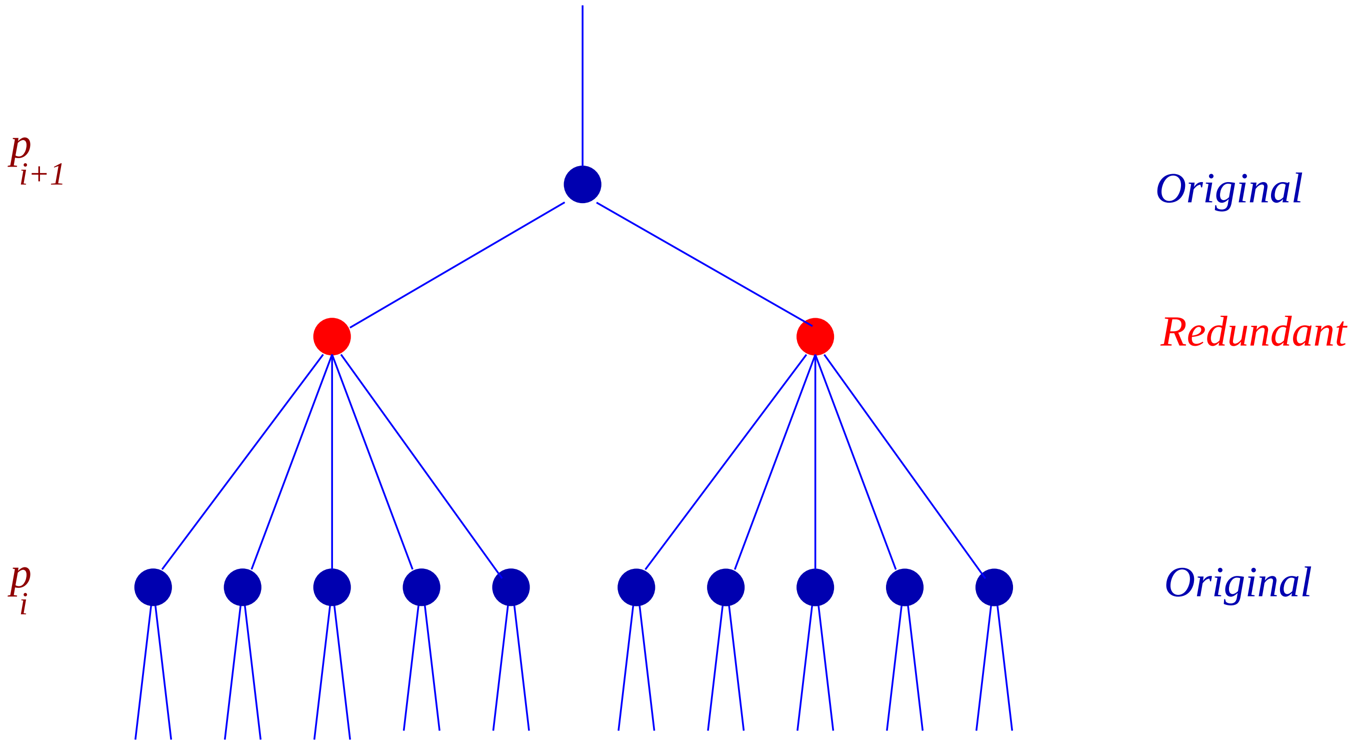
$$p(1-(1-x)^{d-1})^{k-1} < x \qquad \text{for } x \in (0, p).$$

(Luby, Mitzenmacher, Shokrollahi, Spielman, Stemann)

Proof: uses probability theory (martingales, tail inequalities, large deviation results).



Example: $(3, 6)$-regular graph



$p_{i+1} = p_0(1 - (1 - p_i)^5)^2$, so

$$p_0 \left(1 - (1 - p_i)^5\right)^2 < p_i$$

guarantees successful decoding.

General Case

λ_i, ρ_i fraction of edges of degree i on the left and the right hand side.

$\lambda(x) := \sum_i \lambda_i x^{i-1}$ and $\rho(x) := \sum_i \rho_i x^{i-1}$.

Successful decoding for erasure probability p_0 if and only if

$$p_0 \lambda \left(1 - \rho(1 - x)\right) < x$$

for all $x \in (0, p_0)$.

Design of Graphs: Linear Programming

Fix right hand side $\rho(x)$, and find best left hand side $\lambda(x)$ using the condition

$$p_0\lambda\left(1-\rho(1-x)\right) < x$$

on $(0,1)$ using linear programming.

Once best left hand side found, fix left hand side and use dual condition

$$\rho\left(1-p_0\lambda(1-x)\right) > x$$

on $(0,1)$ with linear programming to find best right hand side.

Iterate!

Capacity Achieving Codes

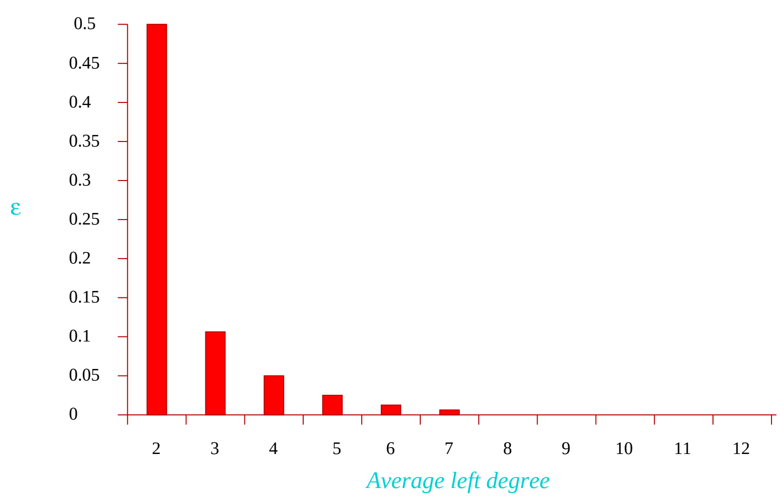
Highly irregular graphs give for any rate R sequences of codes that achieve capacity of erasure channel asymptotically.

Degree structure? Fix design parameter D.

$$\lambda(x) := \frac{1}{H(D)}\left(x + \frac{x^2}{2} + \cdots + \frac{x^D}{D}\right)$$

$$\rho(x) := \exp\left(\mu(x-1)\right)$$

$H(D)$ is harmonic sum $1 + 1/2 + \cdots + 1/D$ and $\mu = H(D)/\left(1 - 1/(D+1)\right)$.



Right Regular Sequences

Fix $a \geq 2$, $n \geq 2$, $\alpha = 1/(a-1)$.

$$\lambda_{a,n}(x) \;:=\; \frac{\sum_{k=1}^{n-1} \binom{\alpha}{k}(-1)^{k+1}x^k}{1 - n\binom{\alpha}{n}(-1)^{n+1}},$$

$$\rho_a(x) \;:=\; x^{a-1}.$$

give codes of rate

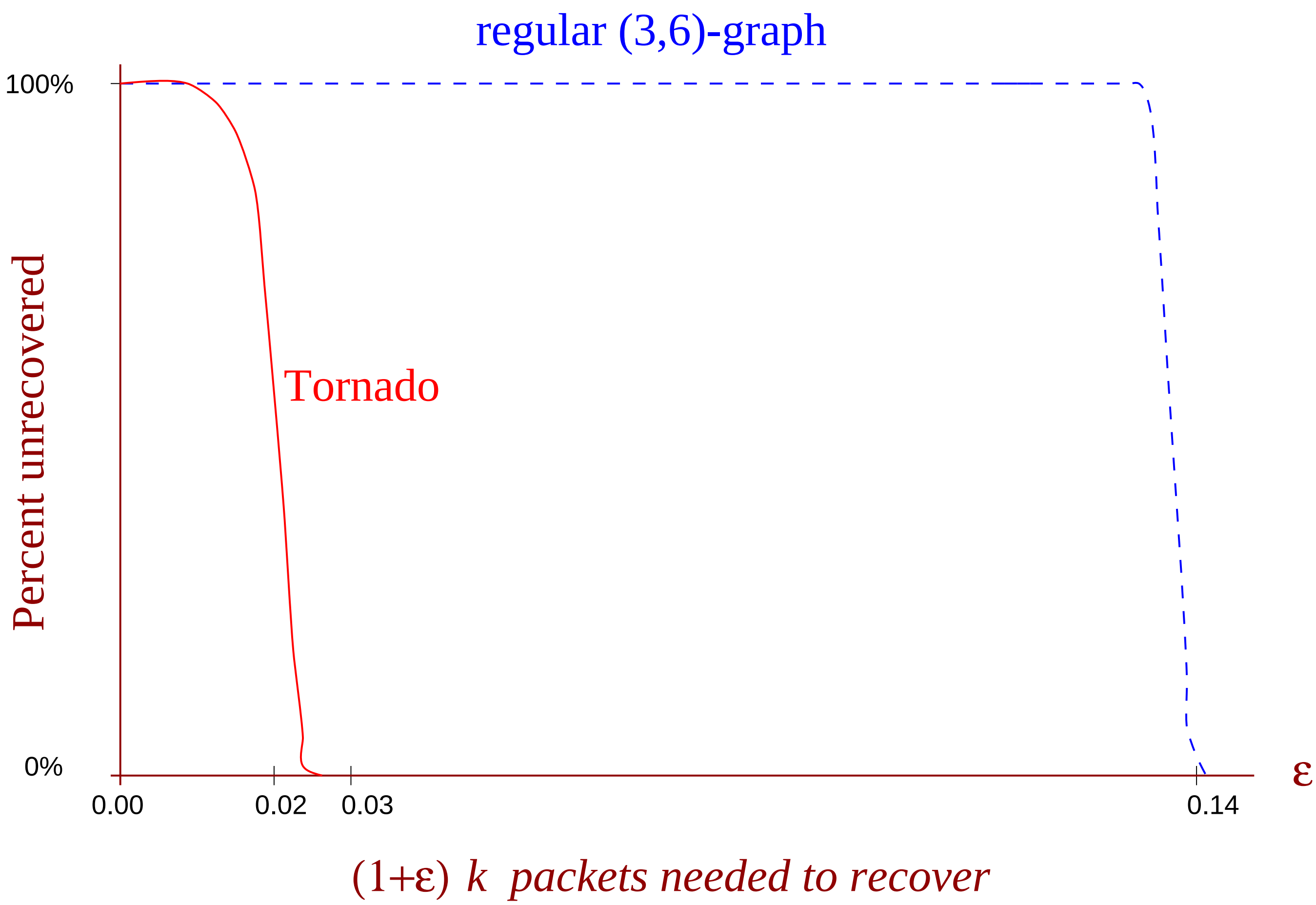
$$1 - \frac{\alpha - \binom{\alpha}{n}(-1)^{n+1}}{\alpha - n\binom{\alpha}{n}(-1)^{n+1}}.$$

In both cases: ε close to channel capacity needs graphs of average degree $\log(1/\varepsilon)$.

Optimal!

(Shokrollahi, 1999)

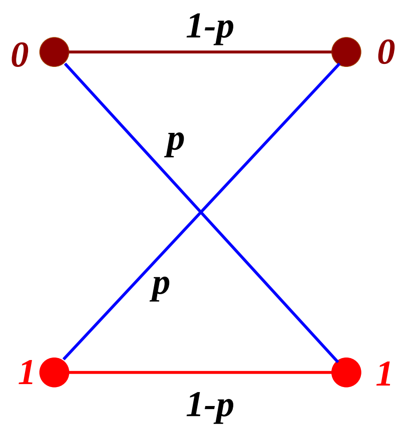
Experiments



640,000 message packets were encoded into 1,280,000 packets using a code with average left degree 8.

Extensions

Binary symmetric channel with hard-decision decoding

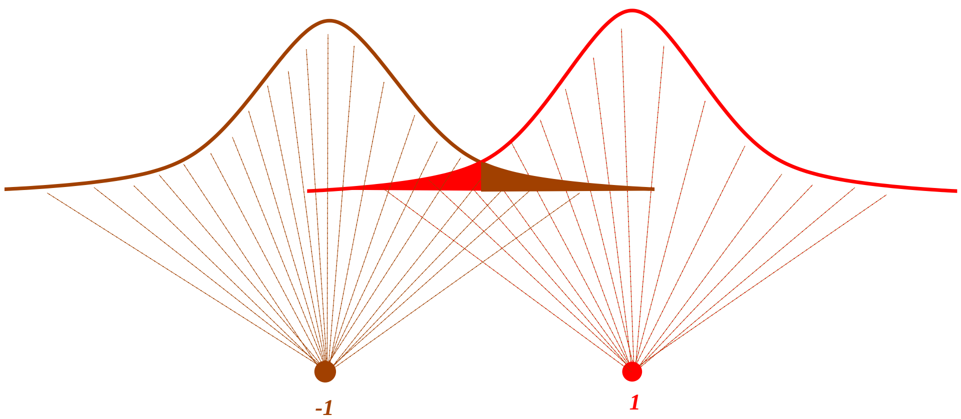


Luby, Mitzenmacher, Shokrollahi, Spielman, STOC 1998

Proof methods were vastly generalized by Richardson & Urbanke.

Extensions

AWGN channel: empirical results

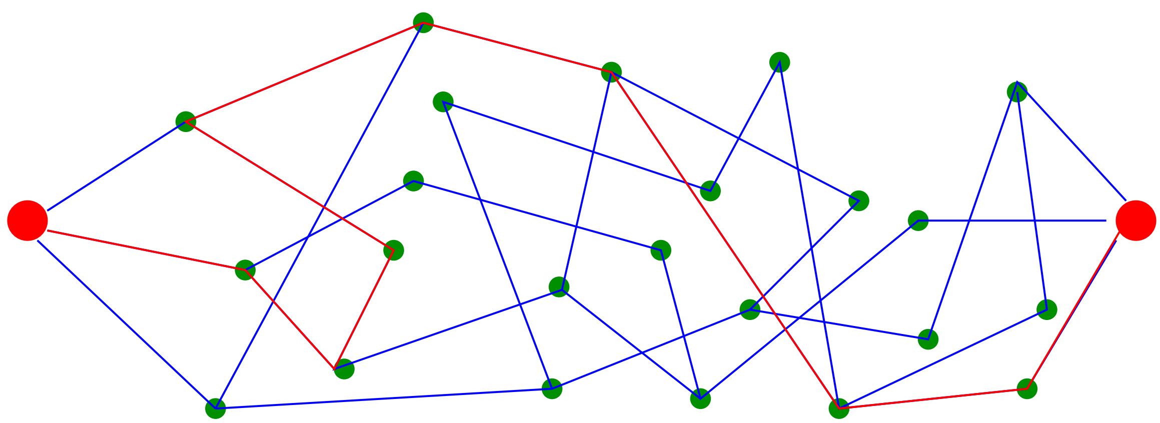


Luby, Mitzenmacher, Shokrollahi, Spielman, ISIT 1998

Applications

Packets in Networks

Data sent in a network is divided into packets which are routed through the network from a sender to a recipient.



Packet Loss

Each packet has an identifier.

Packets can get lost or corrupted.

Corruption is checked via checksums.

Corrupted packets are regarded as lost.

May without loss of generality only concentrate on losses.

Retransmission

In many communication protocols lost packets are re-transmitted.

Process is repeated until all packets are received.

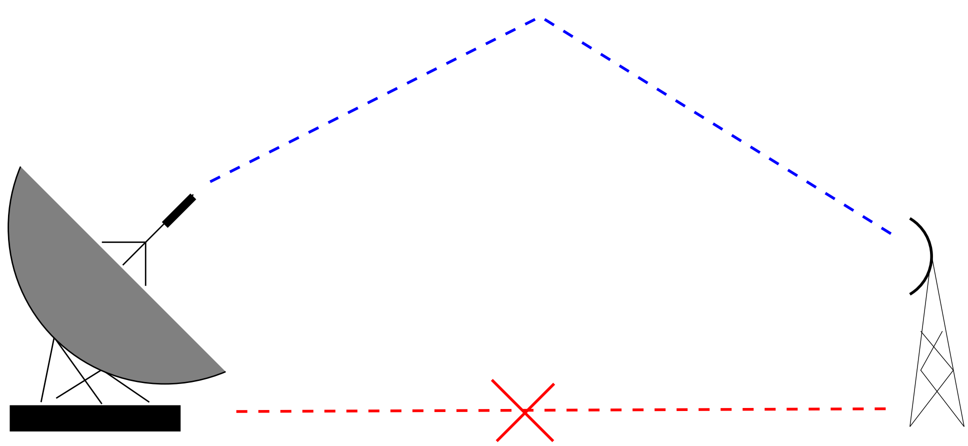
1st round	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
2nd round	3	6	9	13	15	16	17	21	23	27																	
3rd round	6	15	16	23	27																						
4th round	15	23																									
5th round	15																										

Retransmission Protocols

Requires existence of feedback channel.

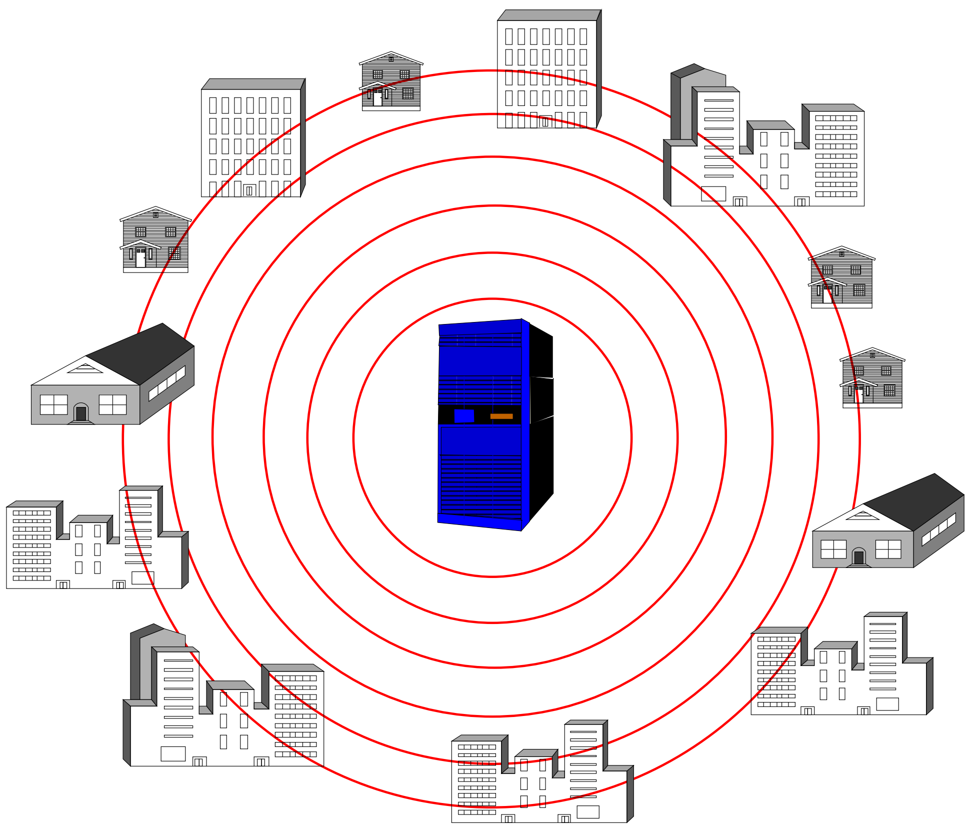
May not exist, or maybe too expensive.

Example: satellite links.



Retransmission Protocols

Not good enough in broadcast application: one server, many clients. Request for retransmission leads to huge server load.



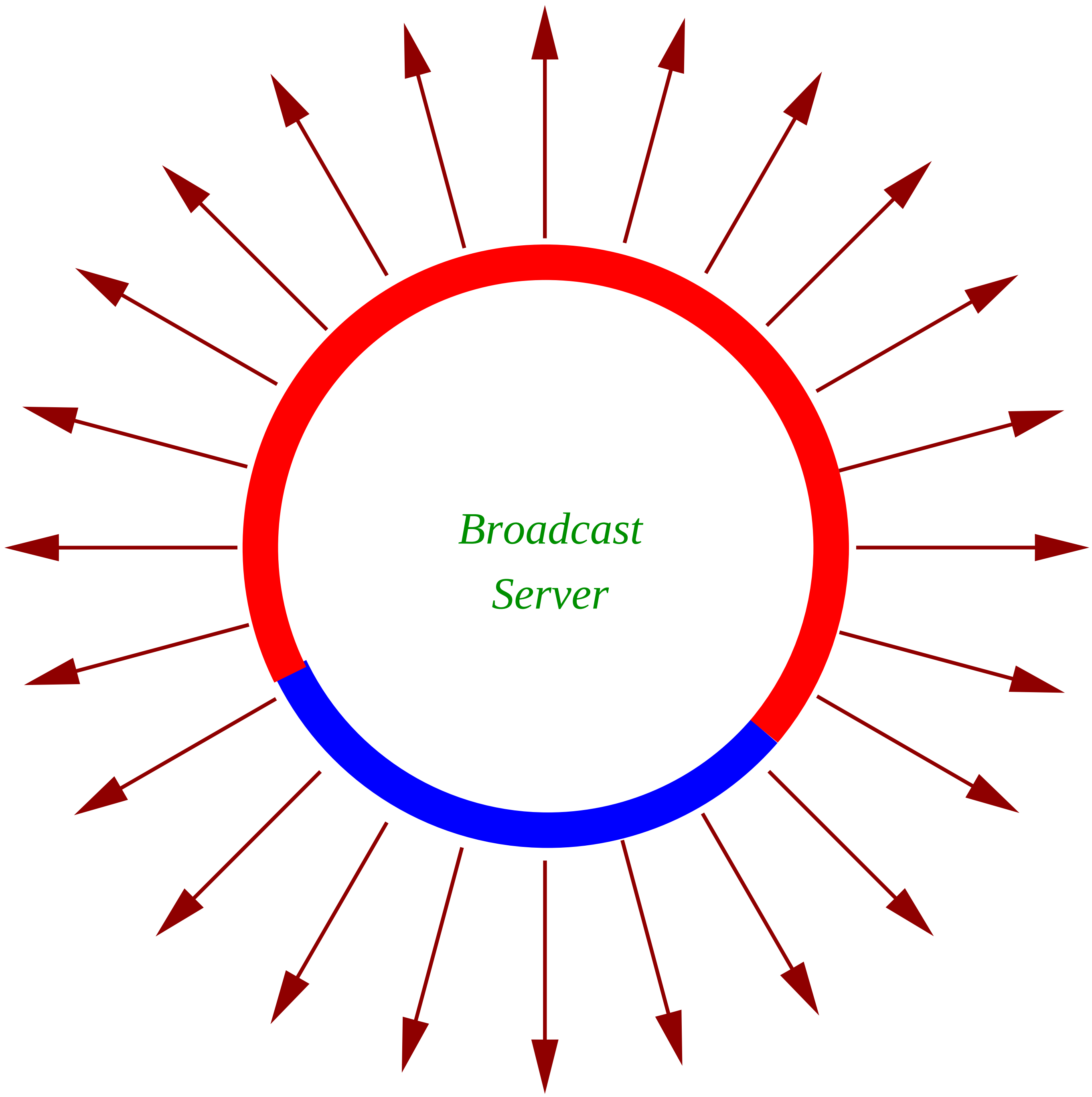
Bulk Data Distribution

Distribution of bulk data to a large number of clients.

Want

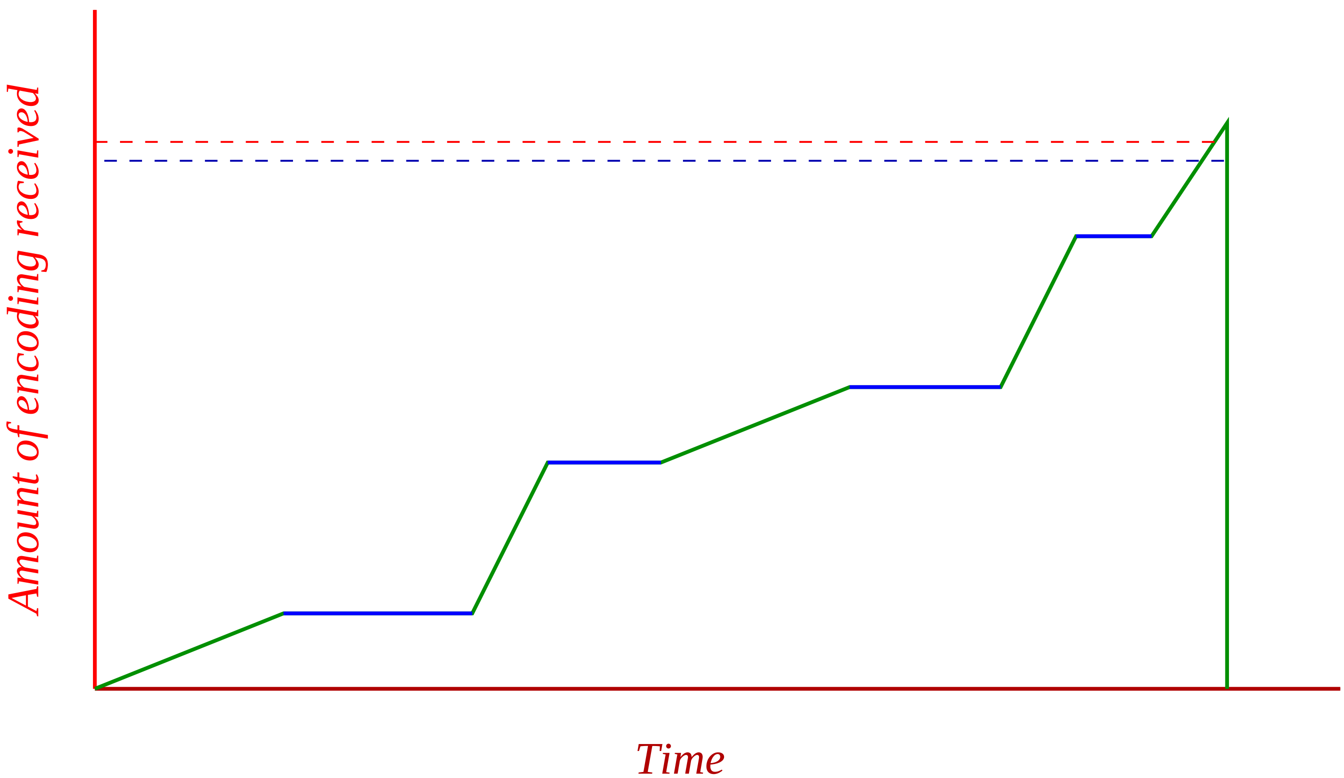
- fully reliable,
- low network overhead,
- support vast number of receivers with heterogeneous characteristics
- users want to access data at times of their choosing and these access times overlap.

Our solution



Our solution

Client joins multicast group until enough of the encoding has been received, and then decodes to obtain original data.



$$ Applications $$

- video and financial information broadcast
- database replication
- popular web site access

Done by

- Tornado codes
- Digital Fountain
- http://www.dfountain.com