Proving the GV Bound Using Graph Theory

We consider the Gilbert graph $G(d)$ with vertex set $V(G) = \mathbb{F}_2^n$ and where two nodes are adjacent if their Hamming distance is less than or equal to d. Then an independent set in this graph is a code of length n and minimum distance at least d; the largest possible size of a code of length n and minimum distance d is the independence number $\alpha(G)$.

The following is a well-known lower bound on the independence number of a graph.

Proposition 1 *Let G be a graph on N vertices and with maximum degree Δ. Then*

$$\alpha(G) \geq \frac{N}{\Delta + 1}.$$

Proof Let I be a maximal independent set. We want to count the number e of edges going between I and $V \backslash I$. On one hand, since I is maximal, every node in $V \backslash I$ is connected to at least one node in I, so that $e \geq |V \backslash I| = n - |I|$. On the other hand, every node in I has at most Δ outgoing edges, so that $e \leq |I|/\Delta$. Putting these two inequalities together gives us the bound. □

Applying this bound to the Gilbert graph with $N = 2^n$ and $\Delta = V(n, d-1) - 1$ (actually, the graph is Δ-regular) immediately gives the GV bound.

Theorem 1 *[2]*

$$\alpha(G(d)) \geq \frac{2^n}{V(n, d-1)}.$$

Using Graph Entropy

Definition 1 *Let G be a graph on N vertices and P be a probability distribution on the vertex set $V(G)$. Then the entropy of G with respect to P is defined as*

$$H(G, P) = \min_{a \in VP(G), a > 0} - \sum_{i=1}^{N} p_i \log a_i.$$

Here $VP(G)$ denotes the *vertex packing polytope* of the graph G, that is, the convex hull in $\mathbb{R}^N$ of the characteristic vectors of the independent sets of G.

When P is taken to be the uniform distribution, we simply talk about the graph entropy $H(G)$.

Let $\alpha(G)$ denote the independence number of the graph G. Then the following theorem provides an upper bound on the graph entropy $H(G)$:

Theorem 2

$$H(G) \geq -\log \frac{\alpha(G)}{N}.$$

The proof is an easy application of the concavity of the logarithm function. We thus have for our Gilbert graph

$$\alpha \geq N2^{-H(G)} = 2^{n-H(G)}.$$

We would like to be able to use this lower bound on α instead of the trivial lower bound of Proposition 1. For this, we need good upper bounds on $H(G)$. The following easy upper bound on $H(G)$ already allows us to retrieve the GV bound.

Proposition 2 *[1]*

$$H(G) \leq \log \chi(G),$$

where $\chi(G)$ denotes the chromatic number of the graph G.

If we use the trivial upper bound $\chi(G) \leq \Delta$ with $\Delta = V(n, d-1) - 1$, we immediately get that

$$H(G) \leq \log V(n, d-1) - 1, \tag{1}$$

so that

$$\alpha \geq \frac{2^n}{V(n, d-1) - 1}.$$

Note that we didn't have to go through the graph entropy to get this result. We could have applied directly the well-known property that

$$\chi(G) \geq N/\alpha(G).$$

In general, if we want to improve on the GV bound $\alpha \geq 2^{n-nH_2(\delta)+o(n)}$ using graph entropy, we will need to have

$$2^{n-H(G)} \geq 2^{n-nH_2(\delta)},$$

i.e., we will need a tighter upper bound than

$$H(G) \leq nH_2(\delta),$$

which is the trivial upper bound of equation (1). To get the Gaborit and Zémor bound $\alpha \geq n2^{n-nH(\delta)+o(n)}$, we will need

$$H(G) \leq nH_2(\delta) - \log n.$$

Can we get this by bounding more carefully the chromatic number of the Gilbert graph?

References

[1] G. Simonyi, "Graph Entropy: a Survey," 1991.

[2] T. Jiang and A. Vardy, "Asymptotic Improvement of the Gilbert-Varshamov Bound on the Size of Binary Codes", *IEEE Trans. Inf. Theory*, vol. 50, no. 8, pp. 1655-1664, Aug. 2004.