The Hadamard transform preserves distances, so fi[rst]
compute

$$H(\mathcal{D}-\mu) = \underbrace{H\mathcal{D}}_{\substack{\text{has been}\\ \text{computed}\\ \text{in Pakzad-Shokrollahi}}} - \underbrace{H\mu}_{\text{can compute, since } \mu \text{ is known}}$$

Once $H(\mathcal{D}-\mu)$ is known, can compute $\|\mathcal{D}-\mu\|_2$ o[...]
in Pakzad-Shokrollahi.

COMPUTING Ω_2 FOR OUR "LDGM" ENSEMBLE

Upto d ones present in every column of G, i.e., every row of G^T.

$G \to k \times n$

$H \to (n-k) \times n$

$H G^T = 0$

Need the number of rows of weight two in H.

$$\underset{\underline{h}}{[1\ 1\ 0\ 0 \cdots\ 0]} \qquad \begin{array}{c} \begin{bmatrix} 1 & \cdots & 0 \\ 0 & \cdots & 1 \\ 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \\ 1 & \cdots & 0 \end{bmatrix}_{n \times k} \\ \underline{g}_1 \quad\quad \underline{g}_k \end{array}$$

Rows of G^T are independent of each other.

$$P(\underline{h}\,\underline{g}_1 = 0) = P(1 \,\&\, 1 \text{ in first two entries of } \underline{g}_1)$$
$$+ P(0 \,\&\, 0 \text{ in " " " " "})$$

$\phi < a$ $\qquad\qquad$ $g'(\cdot) - \rho < 0$

$\frac{1}{1-R} < a \Rightarrow 1-R > \frac{1}{a}$ $\qquad$ $\phi >$

$R < 1 - \frac{1}{a}$

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GAUSSIAN THRESHOLD

① $R \ln\left[\frac{1}{1+e^{-2d/1-R}}\right] < \rho - Cap(\epsilon)\ln(2)$

$$\underbrace{R\left[\ln\left(\frac{1}{1+e^{-2d/1-R}}\right)\right]}_{\text{increasing fn. of } R} < \rho - Cap(\epsilon) . \ln(2)$$

The first condition is always satisfied, see explana[tion] belo[w]

② Matlab reveals that $f(\lambda, \phi)$ is an increasi[ng] function of ϕ, for fixed λ.

For λ fixed, we set

$$\phi'(\lambda) = \arg\max_{1 \le \phi \le \infty} \{f(\lambda, \phi) - \rho < 0\}$$

This gives us that $\phi \le \phi'(\lambda) \quad \forall \lambda \geq$

$$\Rightarrow \phi < \min_{\lambda \geq 0} \phi'(\lambda)$$

$$\Rightarrow R_1 > \frac{1}{\min_{\lambda \geq 0} \phi'(\lambda)}$$

③ Replace $f(\lambda, \phi)$ in ② with $g'(\lambda, \phi)$ and R_1 ...

④ $R > \min\{R_1, R_2\}$.

CLOSER LOOK AT LAST PART OF AWGN SUM

$$e^{\ln b - R \ln q} = e^{\ln\left(\frac{b}{q^R}\right)} = b/q^R$$

(~~From 3 pages before~~)

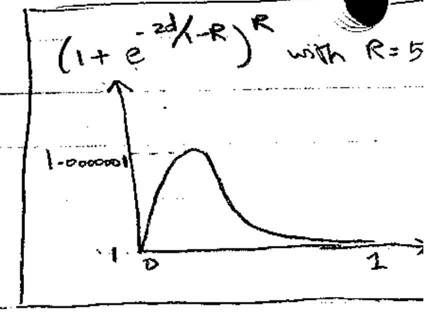
$$q = \frac{1 + e^{-2\lambda}}{2} = \frac{1 + e^{\frac{-2d}{1-R}}}{2}$$

$$\frac{b}{q^R} = \frac{2^{Cap(\epsilon)}}{2^R q^R} = \frac{. \; 2^{Cap(\epsilon)}}{\left(1 + e^{\frac{-2d}{1-R}}\right)^R}$$

$$2^{\frac{1}{2}\log(1+\cancel{SNR}\,\rho)} = e^{-\rho}$$

$$e^{-\rho} = (1+\rho)^{1/2}$$

CHECK -
min-max for AWGN/BSC
Run for low ρ, high SNR



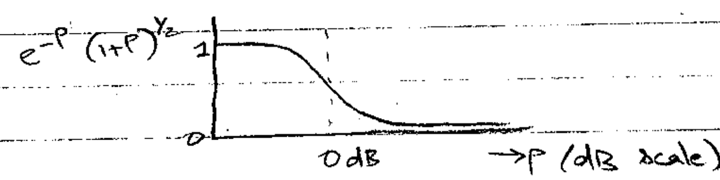
$$e^{-\rho}\frac{b}{q^R} = \frac{e^{-\rho}\,2^{Cap(C)}}{\left(1+e^{-2d/1-R}\right)^R}$$

$$\le e^{-\rho}\,2^{Cap(C)}$$

$$\le e^{-\rho}\,2^{\frac{1}{2}\log_2(1+\rho)}$$

$$= e^{-\rho}(1+\rho)^{1/2}$$

A matlab simul shows that this is always s



Hence the last-part of the AWGN sum converg
to zero everywhere!

MIDDLE PART OF AWGN SUM, 2^{ND} APPROACH:

(From 2 pages before)

$$g'\left(\lambda, \tfrac{1}{1-R}\right) = \frac{d}{a\lambda\tanh\lambda}\ln\left[\left(\frac{d}{1-R}\right)^{\frac{1}{1-R}}\left(\frac{d}{1-R} - a\lambda\tanh\lambda\right)^{\frac{a\lambda\tanh\lambda}{d} - 1}\right.$$

$$\cosh\lambda \cdot \left(\frac{a\tanh\lambda}{e}\right)^{a\lambda\tanh\lambda} \cdot b^{\frac{1}{1-R}} \cdot \frac{1}{(a\lambda\tanh\lambda)^{a\lambda/d\,\tanh\lambda}}$$

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AWGN CHANNEL, $n<k$ CASE

When $n<k$, we should quantize, not channel-c

$$\left[\;\underline{b}^T\;\right]_{1\times k}\left[\;G\;\right]_{k\times n} = \left[\;\underline{x}^T\;\right]_{1\times n}$$

$n<k$

$Cap(\mathcal{C}) \le \frac{1}{2}\log(1+P)$

Lets consider the case that G is full-rank (intuitively, this maximizes the mutual info.)
As $\underline{b}$ varies over all k-tuples, $\underline{x}$ varies over all n-tuples, each n-tuple appearing ? in the "quantizer codebook".
Hence

$$B_W = 2^{k-n}\binom{n}{w}$$

$$b^n \sum_{w=0}^{n} B_w e^{-\rho w} = b^n 2^{k-n} \sum_{w=0}^{n} \binom{n}{w}\left(e^{-\rho}\right)^w$$

$$= b^n 2^{k-n}\left(1+e^{-\rho}\right)^n$$

$$\Pr\left\{ b^n \sum_{w=0}^{n} B_w e^{-\rho w} \ge 2^{n\epsilon} \right\} \ ? \qquad \left(\text{Here } b = \sqrt{\tfrac{e}{2}}\,\frac{2^{Cap(\mathcal{C})}}{2^R}\right)$$

$$b\, 2^{\frac{k}{n}-1}\left(1+e^{-\rho}\right) \overset{?}{\ge} 2^{\epsilon}$$

$$\sqrt{\frac{e}{2}}\,\frac{2^{Cap(\mathcal{C})}}{\cancel{2^R}}\cdot 2^{\cancel{R}-1}\left(1+e^{-\rho}\right) \overset{?}{\ge} 2^{\epsilon}$$

Suppose that $\sqrt{\frac{e}{2}} = 2^{\epsilon}$

$$2^{Cap(\mathcal{C})}\left(\frac{1+e^{-\rho}}{2}\right) \overset{?}{\ge} 1$$

$$1+e^{-\rho} \overset{?}{\ge} \frac{2}{2^{Cap(\mathcal{C})}} \iff e^{\rho} \overset{?}{<} \frac{1}{2^{1-Cap(\mathcal{C})}}$$

$$\rho \overset{?}{<} \ln\left[\frac{1}{2^{1-Cap(\mathcal{C})}-1}\right]$$

Matlab shows that the above never happens $\Rightarrow$ the AWGN, $R>1$ always achieves capacity.

$\phi >$ ___

$\frac{1}{1-R} >$ ___ $\quad 1-R <$

$\boxed{R >}$

$$\phi \ln b + \frac{\lambda \tanh \lambda}{d} \ln\left(\frac{ed\phi}{\lambda \tanh \lambda}\right) + \ln \cosh \lambda$$
$$+ \lambda \tanh \lambda \ln\left(\frac{\tanh \lambda}{e}\right) - \frac{P\lambda \tanh \lambda}{d} < 0$$

$$\ln\left[b^{\phi}\left(\frac{ed\phi}{\lambda \tanh \lambda}\right)^{\frac{\lambda \tanh \lambda}{d}}\right] - \frac{P\lambda \tanh \lambda}{d} + \ln \cosh \lambda$$
$$+ \lambda \tanh \lambda \ln\left(\frac{\tanh \lambda}{e}\right)$$

$$f(\lambda, \phi) < 0, \quad \phi = \frac{1}{1-R}$$

$$b = \sqrt{\frac{e}{2}}\, 2^{\mathrm{cap}(\epsilon) - R}$$

Should hold true for $\phi >$ ___

i.e., $f(\lambda, \phi)$ should be a decreasing fn of ϕ.

$$f(\lambda, R) = \ln\left[b^{\frac{1}{1-R}}\left(\frac{ed}{(1-R)\lambda \tanh \lambda}\right)^{\frac{\lambda \tanh \lambda}{d}}\right] - \frac{P\lambda \tanh \lambda}{d}$$
$$+ \ln \cosh \lambda + \lambda \tanh \lambda \ln\left(\frac{\tanh \lambda}{e}\right) < 0$$

$f(\lambda, R) < 0$ for convergence

Should be a decreasing fn of R.

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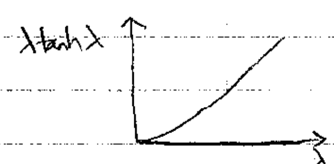
<u>THOUGHTS ON THRESHOLD ANALYSIS</u>:

- For the middle part of the sum,

$$\delta n < w < (1-\delta) n$$

$$\Rightarrow \frac{\delta n}{n-k} < \underbrace{\frac{w}{n-k}}_{\triangleq x \triangleq \frac{\lambda \tanh \lambda}{d}} < \frac{(1-\delta) n}{n-k} \triangleq (1-\delta)\phi$$



As $n, k \to \infty$,
$$\frac{d\delta}{1-R} < \lambda \tanh \lambda < \frac{d(1-\delta)}{1-R}$$

$\frac{1}{\phi} > \max_\lambda f(\lambda)$

$\frac{k}{n} >$ "

$\frac{n}{k} < \frac{1}{\max_\lambda f(\lambda)}$

$x \tanh x \le y$ $y = \lambda \tanh \lambda \le \lambda$

$G_{k \times n}$ 10^6 $\left[\quad H \quad \right]_{n-k \times n}$

$\mathcal{C}$: k dim subsp

$R\uparrow \Rightarrow \frac{k}{n}\uparrow \Rightarrow k\uparrow$ for n fixed

So it is sufficient that the maximization of the threshold over λ be restricted to $0<\lambda<d$

- Consider the term

$$T = \underbrace{\left(\sqrt{\frac{e}{2}}\, 2^{Cap(C)-R}\right)^n}_{T1} \underbrace{\sum_{w=0}^{n} B_w e^{-\rho w}}_{T2}$$

As $R\uparrow$, we expect that $B_w \uparrow$ (since # of codew increases with R)

$\left\{ \begin{array}{l} \mathcal{C}: \text{k-dim subsp of n-dim space} \\ R\uparrow \Rightarrow k/n \uparrow \Rightarrow k\uparrow \text{ for n-fixed} \end{array} \right\}$

For fixed ρ, hence $T2\uparrow$ as $R\uparrow$

" " " , $T1 \downarrow$ as $R\uparrow$

Hence T might not be monotonic!

- For the BSC case, B_w was the # of codewords with weight w in the dual-code.
Hence $B_w \downarrow$ when $R\uparrow$

- Make a scatter plot of R vs. ρ, with a dot indicating that atleast one of the two conditions are satisfi