Let $\eta_1(s,n), \ldots, \eta_n(s,n)$ denote the contents of the n cells in the equiprobable scheme of allocating s particles into n cells.

$P(\xi_{t_1,\ldots,t_m} = 1) = P\{\eta_1(rm,n) \in E, \ldots, \eta_n(rm,n) \in E\}$

$E \rightarrow$ set of even numbers.

Let $P_E(rm,n) \triangleq P(\xi_{t_1,\ldots,t_m} = 1)$

Average # of hypercycles (= Avg. # of non-trivial vectors in the left null space of A)

$$\sum_{m=1}^{T} \binom{T}{m} P_E(rm,n)$$

$$P_E(s,n) \leq (\cosh \lambda)^n \frac{s!}{\lambda^s n^s}, \quad \lambda > 0 \text{ can be chosen arbitrar.}$$

$$P_E(rm,n) \leq (\cosh \lambda)^n \frac{(rm)!}{\lambda^{rm} n^{rm}}$$

<u>CODEWORD</u> <u>STATISTICS</u>:

LDPC ensemble, with max d constraints per codeword

$$\underset{n-k \times n}{H} \, \underline{c} = \underline{0}$$

H has in each row a maximum of d o…
(d ones unif. distributed in n bins)

Hypercycle involving w columns should hav… even no. of ones in all $(n-k)$ coordinate s…

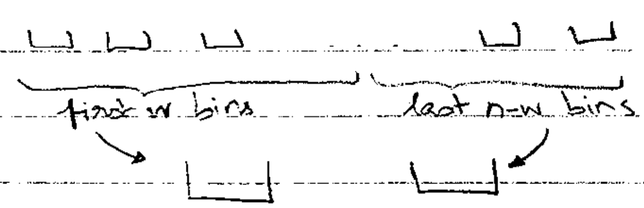
Prob of a hypercycle involving w columns

P_w = Prob { sum of contents of the w entries in row 1 is even, ..., sum of contents of the w entries in row $(n-k)$ is even }

$$= \cancel{\text{Prob}\left\{ \eta_1\left(d, \tfrac{n}{w}\right) \in E \right\}^{(n-k)}}$$

$$= \Pr\left\{ \eta_1(d,n) + \dots + \eta_w(d,n) \in E \right\}^{(n-k)}$$

n bins, d balls.



Pr { ball falling in first equiv. bin } $= \frac{w}{n}$

Repeated trials of a Bernoulli r.v.

$$\text{Bernoulli}\left(\frac{w}{n}, \frac{n-w}{n}\right)$$

$$\text{Prob}(j \text{ items in eq. bin } 1) = \binom{d}{j}\left(\frac{w}{n}\right)^{j}\left(\frac{n-w}{n}\right)^{d-j}$$

Prob(even number of items in eq. bin 1)

$$= \sum_{j=0,2,\dots,d} \binom{d}{j}\left(\frac{w}{n}\right)^{j}\left(\frac{n-w}{n}\right)^{d-j}$$

$$= \sum_{j'=0}^{\lfloor d/2 \rfloor} \binom{d}{2j'}\left(\frac{w}{n}\right)^{2j'}\left(\frac{n-w}{n}\right)^{d-2j'}$$

$$P_w = \left[\sum_{j=0}^{\lfloor d/2 \rfloor} \binom{d}{2j}\left(\frac{w}{n}\right)^{2j}\left(\frac{n-w}{n}\right)^{d-2j} \right]^{n-\alpha n}$$

$$B_w = \binom{n}{w} P_w$$

Need $\displaystyle\sum_{w=0}^{n} B_w e^{-w\rho}$

$$= \sum_{w=0}^{n} \binom{n}{w} \underbrace{\left[\sum_{j=0}^{\lfloor d/2 \rfloor} \binom{d}{2j} \left(\frac{w}{n}\right)^{2j} \left(\frac{n-w}{n}\right)^{d-2j} \right]^{n-\alpha n}}_{\text{this is written in closed-form 1 page later}} e^{-w\rho}$$

Kolchin, pg. 162: Set $p = m/T$, $q = 1-p$. As $T \to \infty$:

$$\binom{T}{m} = \binom{T}{m} p^m q^{T-m} \left(p^m q^{T-m}\right)^{-1}$$

$$= \frac{1}{p^m q^{T-m} \sqrt{2\pi T p q}} \left(1 + o(1)\right)$$

Normal approx to binomial distribution

uniformly in m, $\delta \le m/T \le 1-\delta$

$$\binom{n}{w} = \frac{1}{\left(\frac{w}{n}\right)^{w} \left(\frac{n-w}{n}\right)^{n-w} \sqrt{2\pi \cancel{n} \cdot \frac{w}{\cancel{n}} \cdot \frac{(n-w)}{n}}}$$

DIFFERENT ENSEMBLES:

$H = \Big[\quad \Big]_{(n-k) \times n}$ Consider the case that each column of H contains at most 'd' ones, uniformly distributed according to Kolchin's model.

Vector in the right null-space of H is equivalent to a vector in the left null-space of $A = H^T \in \mathbb{F}_2^{n \times (n-k)}$

Use Kolchin's analysis

$$\text{"}T\text{"} \leftarrow n$$
$$\text{"}n\text{"} \leftarrow n-k$$
$$\text{"}m\text{"} \leftarrow w$$
$$\text{"}r\text{"} \leftarrow d$$
$$\text{"}\alpha = \tfrac{T}{n}\text{"} \leftarrow \frac{n}{n-k} = \frac{1}{1-R}$$

(fixed rate for now, i.e. R is indep. of ρ)

Lemma 3.5.1:

$$\sum_{1 \le w \le \delta n} \underbrace{\binom{n}{w} \underbrace{P_E(dw, n-k)}_{\substack{\text{prob. of hypercycle} \\ \text{involving "}w\text{" rows}}}}_{B_w} e^{-w\rho}$$

$$\le \sum_{1 \le w \le \delta n} \left[\left(\frac{1}{1-R}\right)^{d/2} d^{d/2-1} e^{4d} \delta^{d/2-1} \right]^w e^{-w\rho}$$

$$= \sum_{1 \le w \le \delta n} \left[\left(\frac{1}{1-R}\right)^{d/2} d^{d/2-1} e^{4d} \delta^{d/2-1} e^{-\rho} \right]^w$$

$\le \epsilon$ for some $\delta > 0$.

Lemma 3.5.2:

$$\sum_{(1-\delta)n \le w \le n} \underbrace{\binom{n}{w} P_E(dw, n-k)}_{= B_w} e^{-w\rho} :$$

$$\frac{T!}{m!\,(T-m)!} \qquad \frac{q}{(1-q)^{x}} \qquad \frac{.2}{.8^{x}} \qquad \frac{q^{n}}{(1-q)^{m_0}}$$

$$\sum_{m=T-m_0}^{T} \binom{T}{m} \frac{q^m (1-q)^{T-m}}{q^T} \overset{?}{\le} q^T \Big/ \frac{q^{n-T}}{(1-q)^{m_0}} = q^n \cdot \frac{q^T}{(1-q)^{m_0}}$$

$$= \sum_{m=T-m_0}^{T} \binom{T}{m} \left(\frac{1-q}{q}\right)^{T-m}$$

Let w_0 be an integer such that $w_0 \le n\delta$.

$$\sum_{w=T-w_0}^{T} B_w e^{-\rho w} \le c\,(dn)^{1/2} \left(\frac{q}{(1-q)^{w_0/(\cancel{n}-k-\cancel{x})}} \right)^{\cancel{n}-k-\cancel{x}}$$

$$= c\,(dn)^{1/2} \left(\frac{(1-q)^{-w_0/k}}{q} \right)^{k}$$

$$= c\,(dn)^{1/2} \left(\frac{1}{q^k (1-q)^{w_0}} \right)$$

$$\le \frac{c\,(dn)^{1/2}}{q^{\alpha n} (1-q)^{n\delta}}$$

$$\le c\,(dn)^{1/2}\, q^{\cancel{n}-k} \sum_{w=n-w_0}^{n} \binom{n}{w} \frac{q^w (1-q)^{n-w}}{\cancel{q^n} (1-q)^{w_0}}\, e^{-\rho w}$$

$$= c\,(dn)^{1/2} \frac{q^{-k}}{(1-q)^{w_0}} \underbrace{\sum_{w=n-w_0}^{n} \binom{n}{w} q^w (1-q)^{n-w} e^{-\rho w}}_{\triangleq S_w}$$

$$\frac{S_w}{S_{w+1}} = \frac{\binom{n}{w} \cancel{q^w} (1-q)^{\cancel{n-w}}\, \cancel{e^{-\rho w}}}{\binom{n}{w+1} q^{\cancel{w}+1} \cancel{(1-q)^{n-w-1}}\, e^{-\rho(\cancel{w}+1)}}$$

$$= \frac{\cancel{n!}\,(w+1)\cancel{!}\,\cancel{(n-w-1)!}}{\cancel{w!}\,(n-w)\cancel{!}\,\cancel{n!}} \cdot \left(\frac{1-q}{q}\right) e^{\rho}$$

$$= \frac{w+1}{n-w} \left(\frac{1-q}{q}\right) e^{\rho}$$

(Analysis continued after 3 pages)

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$\binom{2}{0} a^2 b^0 + \binom{2}{1} a b + \binom{2}{2} a^0 b^2$$

<u>ASIDE</u>:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$= \underbrace{\sum_{i=0,2,4,\dots,n} \binom{n}{i} a^i b^{n-i}}_{\triangleq X} + \underbrace{\sum_{i=1,3,\dots,n} \binom{n}{i} a^i b^{n-i}}_{\triangleq Y}$$

$$(b-a)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i a^i b^{n-i}$$

$$= \sum_{i=0,2,4,\dots,n} \binom{n}{i} a^i b^{n-i} - \sum_{i=1,3,\dots,n} \binom{n}{i} a^i b^{n-i}$$

$$= X - Y$$

$$X = \frac{(a+b)^n + (b-a)^n}{2} \qquad \left\{ = \sum_{i=0,2,4,\dots,n} \binom{n}{i} a^i b^{n-i} \right\}$$

For uniform check-node degree LDPC ensemble, from 2 pages before, we have

$$P_w = \left\{ \frac{\left[\frac{w}{n} + \frac{n-w}{n}\right]^d + \left[\frac{n-w}{n} - \frac{w}{n}\right]^d}{2} \right\}^{n-Rn}$$

$$= \left[\frac{1 + \left(\frac{n-2w}{n}\right)^d}{2} \right]^{n-Rn}$$

$$\sum_{w=0}^{n} B_w e^{-wP} = \sum_{w=0}^{n} \binom{n}{w} \left[\frac{1 + \left(\frac{n-2w}{n}\right)^d}{2} \right]^{n-Rn} e^{-wP}$$

$$= \sum_{w=0}^{n} \binom{n}{w} \frac{1}{2^{n(1-R)}} \cdot \left[1 + \left(1 - \frac{2w}{n}\right)^d \right]^{n(1-R)} e^{-wP}$$

Check if this can be computed in close form!

12/11/09

Low density generator matrix (LDGM) ensemble[s]

$\underline{x}_{1\times n} = \underline{b}_{1\times k} A_{k\times n}$

Assume that each of the k columns of A has weight d. Let P_w denote the probabili[ty] that a codeword $\underline{x}$ has weight w.

Let P_1 denote the probability that the first bit of the codeword x_1 is equal to 1.

Since the columns of A are independent of each other, the probability of the x_i bein[g] 1 are independent of each other (and identica[lly] distributed).

$$P_w = P_1^{w} (1-P_1)^{n-w}$$

Computing P_1:

We first compute

$$P(x_1 = 1 \mid w_h(\underline{b}) = j)$$

$$= P\left(\begin{array}{l}\text{having an odd number of ones in} \\ \text{the } j \text{ places where } b_i = 1\end{array}\right)$$

$$= \sum_{i=1,3,\dots,j} \binom{j}{i} p^{i} (1-p)^{j-i}$$

$\hookrightarrow p = \frac{d}{k}$ = prob. of a bit in a column of A being 1

$$= \frac{[\cancel{p} + (1-\cancel{p})]^{j} - [1-2p]^{j}}{2}$$

$$= \frac{1}{2}\left[1 - (1-2p)^{j}\right]$$

Then $p_1 = P(x_1 = 1)$

$$= \sum_{j=1}^{k} \frac{\binom{k}{j}}{2^k} \cdot \frac{1}{2}\left[1 - (1-2p)^j\right]$$

$$= \frac{1}{2^{k+1}} \sum_{j=1}^{k} \binom{k}{j} \left[1 - (1-2p)^j\right]$$

$$= \frac{1}{2^{k+1}} \left[\sum_{j=1}^{k} \binom{k}{j} - \sum_{j=1}^{k} \binom{k}{j} (1-2p)^j \right]$$

$$= \frac{1}{2^{k+1}} \left[(2^k - \cancel{1}) + \cancel{1} - (2-2p)^k \right]$$

$$= \frac{1}{2^{k+1}} \left[\cancel{2^k} - \cancel{2^k}(1-p)^k \right]$$

$$= \frac{1}{2} \left[1 - (1-p)^k \right]$$

$$\sum_{w=0}^{n} B_w e^{-wP} = \sum_{w=0}^{n} \binom{n}{w} P_w e^{-wP}$$

$$= \sum_{w=0}^{n} \binom{n}{w} (P_1 e^{-P})^w (1-P_1)^{n-w}$$

$$= \left[1 - P_1 + P_1 e^{-P} \right]^n$$

$$= \left[\frac{1 + (1-p)^k}{2} + \frac{e^{-P}}{2}\left(1 - (1-p)^k\right) \right]^n$$

$$= \left[\frac{1 + \left(1 - \frac{d}{k}\right)^k}{2} + \frac{e^{-P}}{2}\left(1 - \left(1 - \frac{d}{k}\right)^k\right) \right]^n$$

$$\lim_{x \to \infty} \left(1 + \frac{x}{m}\right)^m = e^x$$

$$\frac{I(\underline{x};\underline{z})}{n} \geq \log\left[\frac{}{\left(\frac{\pi e}{2\rho}\right)^{1/2}} \right.$$

$$\boxed{\text{As } k\to\infty, \quad \sum_{w=0}^{n} B_w e^{-\rho w} = \left[\frac{1+e^{-d}}{2} + \frac{e^{-\rho}}{2}(1-e^{-d}) \right]^n}$$

From 5 pages before,

$$I(\underline{x};\underline{z}) \geq -\log\left\{ \overbrace{\left(\frac{\pi e}{2\rho}\right)^{n/2} \cdot \frac{1}{2^{Rn}} \cdot \sum_{w=0}^{n} B_w e^{-\rho w}}^{\circledast} \right.$$

$$\frac{I(\underline{x};\underline{z})}{n} \geq -\log_2\left\{ \sqrt{\frac{\pi e}{2\rho}} \cdot \frac{1}{2^R}\left[\frac{1+e^{-d}}{2} + \frac{e^{-\rho}}{2}(1-e^{-d}) \right] \right.$$

$\to$ cannot do this! We assumed that $k\to\infty$ which implies that also $n\to\infty$ so above bound will be valid only if $|\circledast|^{1/n} < 1$

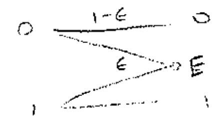
ie. iff

$$\left(\frac{\pi e}{2\rho}\right)^{1/2} \cdot \frac{1}{2^R}\left(\sum_{w=0}^{n} B_w e^{-\rho w}\right)^{1/n} < 1$$

$$\Leftrightarrow \left(\frac{\pi e}{2\rho}\right)^{1/2} \frac{1}{2^R}\left[\frac{1+e^{-d}}{2} + \frac{e^{-\rho}}{2}(1-e^{-d}) \right] <$$

$$\Leftrightarrow \frac{1+e^{-d}}{2} + \frac{e^{-\rho}}{2}(1-e^{-d}) < 2^R\sqrt{\frac{2\rho}{\pi e}}$$

For fixed R, d, this will be true for all $\rho > \rho'(R,d)$, where $\rho'(R,d)$ equates the LHS & RHS of the above bound.



$\frac{\frac{1}{2}\cdot\epsilon}{\epsilon}$

$\frac{\Pr\{X=0 \mid Y=E\}}{\Pr\{X=1 \mid Y=E\}} = \frac{1/2}{1/2} = 1$

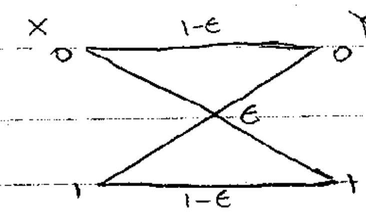
$\frac{1}{0} = \infty$

$$p\,\Delta_{-c_p} + (1-p)\,\Delta_{c_p} = \phi_p$$

$$\sum_{p \in P} c_p \phi_p$$

$P \subseteq R_{>0}$, finite

<u>BSC LLRs</u> :



LLR $\log \frac{\Pr\{X=0 \mid Y\}}{\Pr\{X=1 \mid Y\}}$

$$\boxed{\frac{\Pr\{X=0, Y=0\}}{\Pr\{Y=0\}} = \frac{\frac{1}{2}\cdot(1-\epsilon)}{\cancel{\frac{1}{2}(1-\epsilon)+\frac{1}{2}\epsilon}}}$$

Possible values for BSC:

$$\log \frac{\Pr\{X=0 \mid Y=0\}}{\Pr\{X=1 \mid Y=0\}} = \log \frac{1-\epsilon}{\epsilon}$$

$$\log \frac{\Pr\{X=0 \mid Y=1\}}{\Pr\{X=1 \mid Y=1\}} = \log \frac{\epsilon}{1-\epsilon}$$

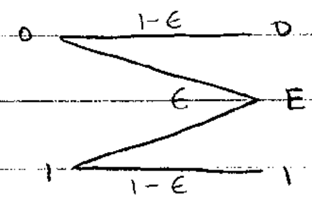
If we assume that the all 1 vector (corresponding to an all zero codeword) was transmitted, then $\log \frac{1-\epsilon}{\epsilon}$ occurs w.p. $1-\epsilon$, and $\log \frac{\epsilon}{1-\epsilon}$ occurs wp. ϵ

$\Rightarrow$ PDF of LLRs is

$$(1-\epsilon)\,\Delta_{\log \frac{1-\epsilon}{\epsilon}} + \epsilon\,\Delta_{\log \frac{\epsilon}{1-\epsilon}}$$

(matches with example 7 in Richardson-Shokrollahi-Urbanke)

BEC LLRs



LLRs:

$$\log \frac{P(X=0 \mid Y=0)}{P(X=1 \mid Y=0)} = \log \frac{1}{0} = \infty$$

$$\log \frac{P(X=0 \mid Y=1)}{P(X=1 \mid Y=1)} = \log 0 = -\infty$$

$$\log \frac{P(X=0 \mid Y=E)}{P(X=1 \mid Y=E)} = \log \frac{1/2}{1/2} = 0$$

Assuming that 0 was transmitted,

$P(Y=0 \mid X=0) = 1-\epsilon$

$P(Y=E \mid X=0) = \epsilon$

$P(Y=1 \mid X=0) = 0.$

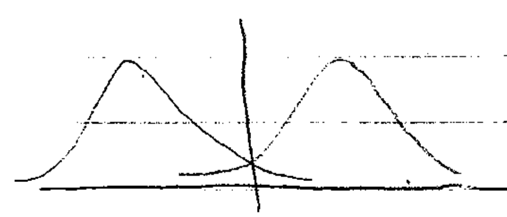
Thus pdf of LLRs:

$(1-\epsilon)\,\Delta_\infty + \epsilon\,\Delta_0$

(matches with example 6 in Richardson–Shokrollahi–Urbanke

AWGN LLRs.

$$\log \frac{P(X=0 \mid Y=y)}{P(X=1 \mid Y=y)}$$



$$P(X=0 \mid Y=y) = \frac{P(X=0, Y=y)}{P(Y=y)} = \frac{P(X=0)\ P(Y=y \mid X=\dots}{P(Y=y)}$$

$= P(X=0)P(Y=y \mid X=0) + P(X=1)\dots$

If 0 is transmitted, any y is possible. Hence pdf of LLRs is a continuous distribution,

$$\sum_{y \in \mathbb{R}} P(Y = y \mid X = 0) \cdot \Delta_{LLR(y)}$$

It is shown in [Richardson - Shokrollahi - Urbanke] that all the above LLR densities are symmetric, i.e.,

$$f(x) = e^{x} f(-x) \quad \text{for} \quad x \in \mathbb{R}$$

Hence we may express the AWGN LLR density as a convex combination of BSC densities

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<u>KOLCHINS</u> <u>ENSEMBLE</u> <u>FOR</u> <u>AWGN</u> (contd from 3 pages before)

$$\sum_{W=T-W_0}^{T} B_W e^{-\rho_W} \le c\,(dn)^{1/2} \frac{q^{-k}}{(1-q)^{W_0}} \sum_{W=n-W_0}^{n} \binom{n}{W} (q e^{-\rho})^{W} (1-q)^{n-W}$$

$$\le c\,(dn)^{1/2} \frac{q^{-k}}{(1-q)^{W_0}} \sum_{W=0}^{n} \binom{n}{W} (q e^{-\rho})^{W} (1-q)^{n-W}$$

$$= c\,(dn)^{1/2} \frac{q^{-k}}{(1-q)^{W_0}} \left[1 - q + q e^{-\rho}\right]^{n}$$

{ this will converge to zero as $n \to \infty$ if $|1-q+qe^{-\rho}| < 1$ (exponential fall dominates $\sqrt{n}$) }

$$\cancel{1} - q + q e^{-\rho} < \cancel{1} \iff e^{-\rho} < 1 \iff e^{\rho} > 1 \iff \rho > 0.$$

<u>Middle</u> <u>part</u> <u>of</u> <u>the</u> <u>sum</u>: $a = \frac{n}{n-k}$, $x = \frac{dw}{n-k}$

$$\binom{n}{w} P_E(dw, n-k)\, e^{-w\rho} = (f(a, x))^{n} \frac{2ad\sqrt{x}}{\sqrt{2\pi\,(ad-x)\,a(n-k)}} e^{-w\rho} (1 + o(1))$$

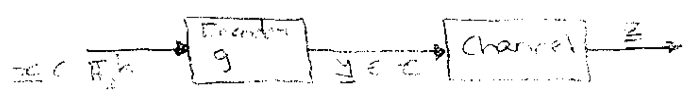
(CONT. AFTER 6 PAGES IN NEXT NOTE-BOOK)

BOUNDS ON RELATIVE ENTROPY

("Info. theory & the central limit theorem" ...
Oliver Thomas Johnson)

Lemma 1.8, pg 10

$$\frac{\log e}{2}\left(\int |p(x) - q(x)|\, dx\right)^2 \le D(p \,\|\, q)$$



$$I(\underline{x}; \underline{z}) = D\left(f(\underline{x}, \underline{z}) \,\|\, f(\underline{x})\, f(\underline{z}) \right)$$

From $f(\underline{x})$, $f(\underline{z})$ in closed form above ...

$$f(\underline{x}, \underline{z}) = f(\underline{x})\, f(\underline{z} \mid \underline{x})$$

$$= \frac{1}{2^k} \cdot N\left(\underline{z} ; g(\underline{x}), \frac{\Sigma}{\rho}\right) \quad \text{(AWGN ch...)}$$

codeword corresponding to $\underline{x}$

$$I(\underline{x}; \underline{z}) \ge \frac{\log e}{2}\left(\int_{x,z} | f(\underline{x}, \underline{z}) - f(\underline{x})\, f(\underline{z}) | \ldots \right.$$

$$= \frac{\log e}{2}\left(\int_{\underline{z}} \sum_{x \in \mathbb{F}_2^k} \left| \frac{1}{2^k} N(\underline{z} ; g(x), \Sigma) - \frac{1}{2^k} \cdot \frac{1}{|\mathcal{C}|} \sum_{i=1}^{|\mathcal{C}|} N(\underline{z} ; \underline{c}_i, \ldots \right.\right.$$

NOTE. $|\mathcal{C}| \ne 2^k$ if rank of g is not full

$$|\mathcal{C}| = \frac{2^k}{|\text{Null sp}(g)|}$$

However, can WOLOG set $|\mathcal{C}| = 2^k$, and let the ...

multiple times over a codeword if the rank is not full.

$$I(\underline{x};\underline{z}) \geq \frac{\log e}{2^{2k+1}} \left(\int_{\underline{z}} \sum_{\underline{x}\in\mathbb{F}_2^k} \Big| N(\underline{z}; g(\underline{x}), I/\rho) - \frac{1}{2^k}\sum_{\underline{x}'\in\mathbb{F}_2^k} N(\underline{z}; g(\underline{x}'), \tfrac{1}{\rho} I \right.$$

$$= \frac{\log e}{2^{2k+1}} \left(\sum_{\underline{x}\in\mathbb{F}_2^k} \int_{\underline{z}} \Big| \frac{1}{(\sqrt{2\pi})^n \left(\frac{1}{\rho^n}\right)^{1/2}} e^{-\frac{1}{2}(\underline{z}-g(\underline{x}))^T \rho I (\underline{z}-g(\underline{x}))} - \frac{1}{2^k}\sum_{\underline{x}'\in\mathbb{F}_2^k} \frac{e^{-\frac{1}{2}(\underline{z}-g(\underline{x}'))^T \rho I (\underline{z}-g(\underline{x}'))}}{(\sqrt{2\pi})^n \left(\frac{1}{\rho^n}\right)^{1/2}} \Big| d\underline{z} \right.$$

$$= \frac{\rho^n \log e}{2^{2k+1}(2\pi)^n} \left(\sum_{\underline{x}\in\mathbb{F}_2^k} \int_{\underline{z}} \Big| e^{-\frac{\rho}{2}|\underline{z}-g(\underline{x})|^2} - \frac{1}{2^k}\sum_{\underline{x}'\in\mathbb{F}_2^k} e^{-\frac{\rho}{2}|\underline{z}-g(\underline{x}')|^2} \right.$$

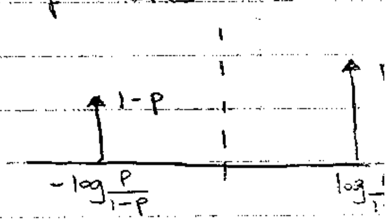
Can we solve this?
What happens in the case of the BSC, when use this bound?

24/11/09

CONVEX COMBINATION OF BSC LLRs

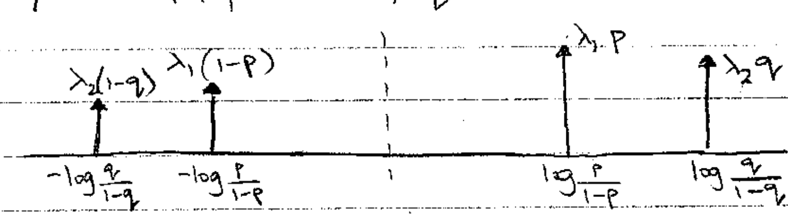
BSC with crossover prob. p : PDF of LLRs is

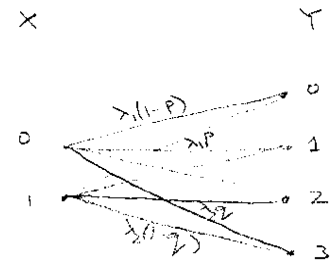
$$\phi_p = p\,\Delta_{\log\frac{p}{1-p}} + (1-p)\,\Delta_{\log\frac{1-p}{p}}$$



Define $\phi = \lambda_1 \phi_p + \lambda_2 \phi_q$

ϕ :





$p(0) = \frac{1}{2}\left[\lambda_1(1-p) + \lambda_1 p\right] = \lambda$

$p(1) = \frac{1}{2}\left[\lambda_1(1-p) + \lambda_1 p\right]$

$p(2) = p(3) = \lambda_2/2$

LLRs for this channel

$$\log \frac{P(X=0|Y=0)}{P(X=1|Y=0)} : \qquad P(X=0|Y=0) = \frac{P(X=0, Y=0)}{P(Y=0)}$$

$$= \frac{P(Y=0|X=0)\,P(X=0)}{P(Y=0|X=0)\,P(X=0) + P(Y=0|X=1)\,P(X=1)}$$

$$= \frac{\lambda_1(1-p)\,\frac{1}{2}}{\lambda_1(1-p)\frac{1}{2} + \lambda_1 p\,\frac{1}{2}}$$

$$P(X=1|Y=0) = \frac{P(X=1, Y=0)}{P(Y=0)} = \frac{\lambda_1 p\,\frac{1}{2}}{\lambda_1 p\frac{1}{2} + \lambda_1(1-p)\frac{1}{2}} = p$$

$$\log \frac{P(X=0|Y=0)}{P(X=1|Y=0)} = \log \frac{1-p}{p} = \mathcal{L}_1$$

$$\log \frac{P(X=0|Y=1)}{P(X=1|Y=1)} = \log \frac{p}{1-p} = \mathcal{L}_2$$

Given that 0 was transmitted $\mathcal{L}_1$ occurs with prob. $P(Y=0|X=0) = \lambda_1(1-p)$, and $\mathcal{L}_2$ with prob $P(Y=1|X=0) = \lambda_1 p$.
– channel is symmetric for $Y=2,3$, giving the LLR ϕ!

NOTE: Need to set $\lambda_1 + \lambda_2 = 1$ to obtain a valid PDF.

$\underline{a} = H \underline{b}$

$\sum a_i^2 = \sum b_i^2$

$$P(Y/X) = \begin{bmatrix} \lambda_1(1-p) & \lambda_1 p & (1-\lambda_1)(1-q) & (1-\lambda_1 \\ \lambda_1 p & \lambda_1(1-p) & (1-\lambda_1)q & (1-\lambda_1)(1 \end{bmatrix}$$

Lemma 4 of Pakzad & Shokrollahi is valid f this channel (being discrete), but the appr value of the capacity of this channel needs to used.

① Find the capacity of this 2x4 channel.

② Express $\mathbb{E}[H(Z)]$ in terms of B_w. (need a bound on $\mathbb{E}[H(Z')]$)

Analysis of $\mathbb{E}[H(Z)]$: Let D be the distributio induced on $\mathbb{F}_4^n$ by the output of the channel.

$$H(D) \geq -\log_2 \left(\sum_{\underline{u} \in \mathbb{F}_4^n} P_{\underline{u}}^2 \right)$$

$P_{\underline{u}} \to$ prob of c vector $\underline{u} \in \mathbb{F}_4^n$ a to distribution

Let H be the $4^n \times 4^n$-Hadamard matrix. Defi

$$q_{\underline{v}} = \sum_{\substack{\underline{u} \in \mathbb{Z}_4^n \\ \langle \underline{u}, \underline{v} \rangle = 1,3}} P_{\underline{u}}$$

$$1 - 2q_{\underline{v}} = 1 - q_{\underline{v}} - q_{\underline{v}} = \sum_{\substack{\underline{u} \in \mathbb{Z}_4^n \\ \langle \underline{u}, \underline{v} \rangle = 0,2}} P_{\underline{u}} - \sum_{\substack{\underline{u} \in \mathbb{Z}_4^n \\ \langle \underline{u}, \underline{v} \rangle = 1,3}} P_{\underline{u}}$$

$$= \sum_{\underline{u} \in \mathbb{Z}_4^n} (-1)^{\langle \underline{u}, \underline{v} \rangle \bmod 4} P_{\underline{u}} = \sum_{\underline{u} \in \mathbb{Z}_4^n} (-1)^{\langle \underline{u}, \underline{v} \rangle \bmod 2}$$

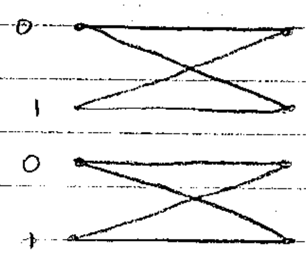
Claim

$$\stackrel{\downarrow}{=} \cancel{\sum_{\underline{u}' \in \mathbb{F}_2^{2n}} (-1)^{\langle \underline{u}', \underline{v}' \rangle \bmod 2} P_{\underline{u}'}}$$

binary inner pro

$$= \sum_{\underline{u} \in \mathbb{Z}_4^n} (-1)^{\langle \underline{u} \bmod 2,\ \underline{v} \bmod 2 \rangle} P_{\underline{u}}$$

How many vectors $\underline{x} \in \mathbb{Z}_4^n$ are equal mod 2 to let $\underline{a} = \underline{u} \bmod 2$. To obtain a vector equal to $\underline{a}$ I can pick one among 2 values for each of the n coord one of these will be $\underline{a}$ itself.



think of o/p as $\mathbb{F}_2^2 = \{00, 01, 10,$
Can we write the noise as ad-
noise in this case?

1^{st} stage: Rep. code

$$0 \mapsto 00$$
$$1 \mapsto 11$$

2^{nd} stage: BSCs with crossover probs z_1 & z_2

$$0_{\mathbb{F}_2} \to 1_{\mathbb{F}_4} \equiv 00_{\mathbb{F}_2} \to 01_{\mathbb{F}_2} \equiv (1-z_1) z_2 = \lambda_1 p$$

$$0_{\mathbb{F}_2} \to 0_{\mathbb{F}_4} \equiv 00 \to 00 \equiv (1-z_1)(1-z_2) = \lambda_1 (1-p) -$$

$$0_{\mathbb{F}_2} \to 2_{\mathbb{F}_4} \equiv 00 \to 10 \equiv z_1 (1-z_2) = (1-\lambda)(1-q) -$$

$$0_{\mathbb{F}_2} \to 3_{\mathbb{F}_4} \equiv 00 \to 11 \equiv z_1 z_2 = (1-\lambda_1) q - (4)$$

$$\frac{①}{②} : \frac{z_2}{1-z_2} = \frac{p}{1-p} \Rightarrow z_2 - \cancel{p z_2} = p - \cancel{p z_2}$$
$$z_2 = p$$

$$① : (1-z_1)\cancel{p} = \lambda_1 \cancel{p} \Rightarrow z_1 = 1 - \lambda_1$$

$$\frac{③}{④} : \frac{1-z_2}{z_2} = \frac{1-q}{q} \Rightarrow z_2 = q$$

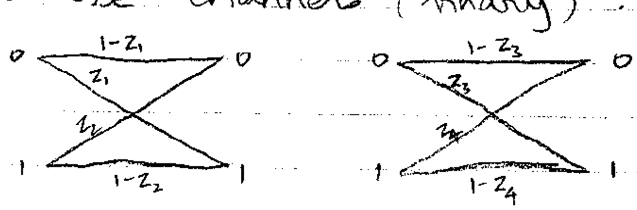
$$④ : z_1 \cancel{q} = (1-\lambda_1)\cancel{q} \Rightarrow z_1 = 1 - \lambda_1$$

Need $p = q$ for above to work

$$1_{\mathbb{F}_2} \to 0_{\mathbb{F}_4} : 11_{\mathbb{F}_2} \to 00_{\mathbb{F}_2} \equiv z_1 z_2 = \lambda_1 p$$

Hence need $\lambda_1 = 1/2$ for the above to work.

Choosing BSCs for the 2^{nd} stage makes it too restric-
Choose channels (binary):



$0_{\mathbb{F}_2} \to 1_{\mathbb{F}_4} \equiv 00_{\mathbb{F}_2} \to 01_{\mathbb{F}_2} \equiv (1-z_1)z_3 = \lambda_1 p$

$0 \to 0 \equiv 00 \to 00 \equiv (1-z_1)(1-z_3) = \lambda_1(1-p)$

$0 \to 2 \equiv 00 \to 10 \equiv z_1(1-z_3) = (1-\lambda_1)(1-q)$

$0 \to 3 \equiv 00 \to 11 \equiv z_1 z_3 = (1-\lambda_1) q$

Solving the above will give us

$$z_3 = p = q$$
$$z_1 = 1-\lambda_1$$

$1_{\mathbb{F}_2} \to 0_{\mathbb{F}_4} \equiv 11 \to 00 \equiv z_2 z_4 = \lambda_1 p$

$1 \to 1 \equiv 11 \to 01 \equiv z_2(1-z_4) = \lambda_1(1-p)$

$1 \to 2 \equiv 11 \to 10 \equiv (1-z_2) z_4 = (1-\lambda_1) q$

$1 \to 3 - 11 \to 11 \equiv (1-z_2)(1-z_4) = (1-\lambda_1)(1-q)$

$\frac{5}{6} \Rightarrow z_4 = p \qquad \frac{7}{8} \Rightarrow z_4 = q$

$\Rightarrow \cancel{p} z_2 = \lambda_1 \cancel{p} \Rightarrow z_2 = \lambda_1$

$p = q$ is useless, since it gives us a resulta LLR with only two delta function!

Assume that we are working over $\mathbb{F}_4^n$, since $\mathbb{F}_2^n \subseteq \mathbb{F}_4^n$. Code $\mathcal{C} \subseteq \mathbb{F}_2^n \subseteq \mathbb{F}_4^n$. Regard $\mathcal{C}$ as code over $\mathbb{F}_4$.